Riemannian Geometry Framed as a Generalized Lie Algebra to Incorporate General Relativity with Quantum Theory, I

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Abstract: This paper reframes Riemannian geometry (RG) as a generalized Lie algebra allowing the equations of both RG and then General Relativity (GR) to be expressed as commutation relations among fundamental operators. We begin with an Abelian Lie algebra of n "position" operators, X, whose simultaneous eigenvalues, y, define a real n-dimensional space R(n). Then with n new operators defined as independent functions, X'(X), we define contravariant and covariant tensors in terms of their eigenvalues, "y", on a Hilbert space representation. We then define n additional operators, D, whose exponential map is to translate X as defined by a noncommutative algebra of operators (observables) where the "structure constants" are shown to be the metric functions of the X operators to allow for spatial curvature resulting in a noncommutativity among the D operators. The D operators then have a Hilbert space position-diagonal representation as generalized differential operators plus an arbitrary vector function A(X), which, with the metric, written as a commutator, can express the Christoffel symbols, and the Riemann, Ricci, and other tensors as commutators in this representation thereby framing RG as a Generalized Lie algebra (GLA). Traditional GR is obtained in the position diagonal representation of this noncommutative algebra of 2n+1 operators. Our motivation was suggested by the fact that Quantum Theory (QT), Special Relativity (SR), and the Standard Model (SM) are framed and well-established in terms of Lie algebras. But GR, while also well-established, is framed in terms of nonlinear differential equations for the space-time metric and space-time variables. This GLA provides a more general framework for RG to support an integration of GR, QT, and the SM by generalizing Lie algebras as described. The gauge transformations that here support the SM are altered by the inclusion of the metric. Other non-trivial consequences are discussed such as a generalized uncertainty principle and that the Dirac γ matrices now are position dependent altering the Dirac equation and system Lagrangian. Finally, we initially explore the determination of possible asymmetric components of the metric.

Keywords: Riemannian Geometry, Lie algebra, Quantum Theory, metric, Heisenberg algebra, General Relativity, Standard Model, Uncertainty Principle.

1. Introduction

Lie algebras, and the Lie groups which they generate, have played a central role in both mathematics and theoretical physics since their introduction by Sophius Lie in 1888 [1]. Both relativistic quantum theory (QT) and the gauge algebras of the phenomenological standard model (SM) of particles and their interactions are framed in terms of observables which form Lie algebras and are firmly established [2,3,4,5]. The Heisenberg Lie algebra (HA) among (generalized) momentum and position operators, [D, X] gives the foundational structure of QT and has applications in mathematics in studies related to Fourier transforms and harmonic analysis [6,7,8,9]. Likewise, in QT one has the Poincare symmetry Lie algebra (PA) of space-time observables whose representations define free particles.

But the theory of gravitation as expressed in Einstein's general theory of relativity (GR), although also firmly established, is formulated in terms of a Riemannian geometry (RG) of a curved space-time where the metric is determined by nonlinear differential equations from the distribution of matter and energy [10,11]. In GR there are no operators representing observables, and thus no commutation rules to define Lie algebras, and thus no representations of such algebras. The observables in GR are (a) the positions of events in space-time, and (b) the metric function of position in space-time (and its derivatives) which define the distance between events, and which define the curvature of space-time. Thus, QT and GR are expressed in totally different mathematical frameworks and their merger into a single theory has been a central problem in physics for over a century. However, the space-time events in QT are the eigenvalues of the space-time operators which are an essential part of the HA which also contains the Minkowski metric which defines the associated translated distance when space-time is not curved. If the associated space were curved, one would have a metric that was a function of the position in space-time. Such a generalized HA would no longer allow closure as a traditional Lie algebra but rather closure in the enveloping algebra of analytic functions of the basis elements of the Lie algebra.

This led us to consider a Generalized Lie Algebra (GLA) generalizing the framework of a Lie algebra, with n space-time operators, X^μ, and n corresponding operators D^μ, which by definition are to execute infinitesimal translations in the associated representation space of the X^μ eigenvalues, y^μ ($\mu = 0$, 1, n-1). The X^μ are to form an Abelian algebra whose eigenvalues represent a "space-time" manifold of four or a larger number of dimensions as the associated X eigenvalues are simultaneously measurable. But we allow the space-time to be curved so the corresponding D^μ operators will not in general commute as the translations in such a curved space can interfere with each other. We found that this approach automatically generalized the HA "structure constants" to be the Riemann metric thus allowing the metric to be a function of the position operators, X, in the algebra [12]. This generalizes the concept of a Lie algebra to allow for "structure constants" that are functions of the X operators in the algebra and thus are no longer constants except approximately in small neighborhoods.

This paper first formally reframes RG [13] as a GLA including the HA. We show that the fundamental concepts in RG such as the coordinate transformations, contravariant and covariant tensors, Christoffel symbols, Riemann and Ricci tensors, and the Riemann covariant derivative can now all be expressed in terms of commutation relations among these fundamental operators. This framework is reminiscent of contractions of Lie algebras where the structure constants as functions are modified to vary smoothly among different algebras based upon certain external parameters [14, 15, 16, 17, 18, 19] but not in the algebra itself as we propose. In a similar way, our algebra allows the structure constants to be dependent upon the X operators in the algebra so that RG is retrieved as a position-diagonal representation of the algebra as one moves over the Riemann manifold of X eigenvalues.

2. Riemannian Geometry Framed as a Noncommutative Algebraic Geometry of Observables

We begin by defining a purely mathematical structure devoid of QT, SM, & GR content. Consider a set of n independent linear self-adjoint operators, X^μ, which form an Abelian Lie algebra of order n, where

$[X^\mu, X^\nu] = 0$ and where $\mu, \nu = 0, 1, 2, \dots (n-1)$. (2.1)

It is assumed that the units of measurement are the same for the eigenvalues of all X variables. Consider a Hilbert space of square integrable complex functions $|\Psi\rangle$ as a representation space for this algebra where a scalar product is used to normalize the vectors to unity: $\langle\Psi|\Psi\rangle = 1$. The simultaneous eigenvectors of the Abelian Lie algebra can be written as the outer product of the X^μ eigenvectors with the Dirac notation:

$|y^0\rangle\, |y^1\rangle |y^2\rangle \dots |y^{n-1}\rangle = |\, y^0, y^1, y^2, \dots y^{n-1}\rangle = |y\rangle$ (2.2)

where the eigenvalues y^μ label the associated eigenvectors $|y\rangle$ of the X^μ operators where we use the notation

$X^\mu|y\rangle = y^\mu|y\rangle$ where the y^μ are real numbers defining the Hilbert manifold. (2.3)

These independent real variables y^μ can be thought of as the coordinates (or basis vectors) of an n-dimensional space R_n since each set of values defines a point in R_n. Let the eigenvectors be normalized to be orthonormal with the scalar product:

$\langle y_a|y_b\rangle = \delta(y^0{}_a - y^0{}_b)\, \delta(y^1{}_a - y^1{}_b) \dots \delta(y^{n-1}{}_a - y^{n-1}{}_b)$. (2.4)

Let the decomposition of unity:

$1 = \int_{dy} |y\rangle\langle y|$ (2.5)

project the entire space onto the basis vectors $|y\rangle$ where $\langle y|$, using Dirac notation, is the dual vector to $|y\rangle$. A general vector in the representation (Hilbert) space of this Lie algebra can then be written as

$|\Psi\rangle = \int_{dy} |y\rangle\langle y|\Psi\rangle = \int_{dy} \Psi(y)\, |y\rangle$, (2.6)

where the function $\Psi(y)$ gives the "components" of the abstract vector $|\Psi\rangle$ on the basis vectors $|y\rangle$. Thus

$\langle\Psi|\Psi\rangle = 1 = \int_{dy} \langle\Psi|y\rangle\langle y|\Psi\rangle = \int_{dy} \Psi^*(y)\Psi(y)$. (2.7)

Now consider another set of n linear operators, X'^μ, which are independent analytic functions, $X'^\mu(X^\mu)$, of the X^μ operators also forming an Abelian Lie algebra on the same representation space for this algebra where it follows that:

$[X'^\mu, X'^\nu] = 0$. (2.8)

Let the X'^μ have eigenvectors $|y'\rangle$ and eigenvalues y'^μ given by

$X'^\mu|y'\rangle = y'^\mu|y'\rangle$ where y'^μ are real numbers . (2.9)

The same orthonormality and decomposition of unity also obtain for the $|y'\rangle$ vectors which are also to be a complete basis for the space $|\Psi\rangle$. Then we can let the $X'^\mu(X^\nu)$ act to the left on the dual vector $\langle y'|$ and act to the right on the vector $|y\rangle$ as

$\langle y'|\, X'^\mu|y\rangle = \langle y'|\, X'^\mu(X^\nu)|y\rangle$ to give (2.10)

$y'^\mu\langle y'|y\rangle = y'^\mu(y)\langle y'|y\rangle$. (2.11)

Thus, the eigenvalues $y'^\mu = y'^\mu(y)$ give the transformation from the y coordinates to the y' coordinates if the Jacobian does not vanish $|\partial y'^\mu/\partial y^\nu| \neq 0$ which we require to be the case. Thus, the operator functions $X'^\mu(X^\mu)$ define a coordinate transformation in R_n between the eigenvalues (coordinates) y and the eigenvalues y' (transformed coordinates) that define R_n. Then the set of n real variables y^μ and the alternative variables y'^{μ} both can be interpreted as specifying the coordinates of points in this n-dimensional real space R_n with coordinate transformations given by the functions

$$y'^{\mu} = y'^{\mu}(y). \qquad (2.12)$$

It now follows that:

$$dy'^{\mu} = (\partial y'^{\mu}/\partial y^{\nu})\, dy^{\nu} \qquad (2.13)$$

and any set of n functions $V^{\mu}(y)$ that transform as the coordinates,

$V'^{\mu}(y') = (\partial y'^{\mu}/\partial y^{\nu})\, V^{\nu}(y)$ is to be called a contravariant vector. (2.14)

The upper (contravariant) indices are normally taken as the variables that are normally measured while the lower (covariant) indices are obtained by lowering the index with the metric as shown below. We use the summation convention for repeated identical indices. The derivatives $\partial/\partial y^{\nu}$ transform as

$$\partial/\partial y'_{\mu} = (\partial y^{\nu}/\partial y'^{\mu})\, \partial/\partial y^{\nu} \qquad (2.15)$$

and any such vector $V_{\mu}(y)$ which transforms in this manner as

$V'_{\mu}(y') = (\partial y^{\nu}/\partial y'^{\mu})\, V_{\nu}(y)$ is defined as a covariant vector. (2.16)

Upper indices are defined as contravariant indices while lower indices are covariant indices. Functions with multiple upper and lower indices that transform as the contravariant and covariant indices just shown are defined as tensors of the rank of the associated indices.

One would like to have transformations that translate one in the R_n space of the operators X (and thus their eigenvalues y). We define a new additional set of n operators, D^{μ}, that translate a point an infinitesimal distance, ds, in the R_n space respectively in each corresponding directions y^{μ} by using the transformation generated by the D^{μ} elements of the algebra via the exponential map with transformations:

$$G(ds, \eta) = \exp(ds\, \eta_{\mu} D^{\mu}/b) \qquad (2.17)$$

In this transformation η_{μ} is defined to be a unit vector in the y space, b is an unspecified constant, and ds is defined to be the distance moved in the direction η_{μ} as defined below. Then

$$X'^{\lambda} = G\, X^{\lambda}\, G^{-1}. \qquad (2.18)$$

By taking the translated distanace ds to be infinitesimal, then one gets

$$\begin{aligned} X'^{\lambda} = X^{\lambda}(s+ds) &= \exp(ds\, \eta_{\mu} D^{\mu}/b)\ \ X^{\lambda}(s)\ \ \exp(-ds\, \eta_{\nu} D^{\nu}/b) \\ &= (1 + ds\, \eta_{\mu} D^{\mu}/b)\ X^{\lambda}(s)\ (1 - ds\, \eta_{\nu} D^{\nu}/b) \\ &= X^{\lambda}(s) + ds\, \eta_{\mu} [D^{\mu}, X^{\lambda}]/b\ + \text{higher order in } ds., \end{aligned} \qquad (2.19)$$

Thus, the commutator $[D^{\mu}, X^{\lambda}]$ defines the way in which the transformations commute (interfere) with each other in executing the translations in keeping with the theory of Lie algebras and Lie groups although in general the D & X may not close as a standard Lie algebra. If the space is Euclidian (flat) then there is no dependence of the commutator upon location, and thus there is no interference among the D^{μ}. Then $[D^{\mu}, X^{\lambda}]$ can be normalized to $I\, \delta_{\pm}^{\mu\lambda}$ (since D^{μ} is defined to translate X^{μ}) thus:

$[D^{\mu}, X^{\lambda}] = I\, \delta_{\pm}^{\mu\lambda} = b\, \delta_{\pm}^{\mu\lambda}$ and the space is Euclidian (flat) (2.20)

where $\delta_{\pm}$ is the diagonal n x n matrix with ±1 on the diagonal with off-diagonal terms zero. The units of the constant b then are complementary to the units of the associated y eigenvalues since the product of the X and D eigenvalues must give the units of the constant b so that dimensional units balance for equation (2.20). This is the customary Heisenberg Lie algebra with structure constants $I\delta_{\pm}^{\mu\lambda}$ and with $[D^{\mu}, D^{\lambda}] = 0$ for $\mu \neq \lambda$. The additional operator, I, is to commute with all elements and by definition has a single eigenvalue b, and is needed to close the basis of the Lie algebra which now is of dimension 2n+1. Thus, confirming that the distance is ds:

$dX^{\lambda}(s) = ds\, \eta_{\mu}\, b/b\ \delta_{\pm}^{\mu\lambda} = ds\, \eta^{\lambda} +$ higher order terms in ds. (2.21)

We now wish to allow for curvature in the space R_n of the X eigenvalues. Thus the [D, X] commutator is now allowed to be dependent upon the operators X and can vary from point to point in the non-Euclidian space. We define the functions $g^{\mu\nu}(X)$ as generalized structure functions (no longer constants) as:

$$[D^{\mu}, X^{\nu}] = b\, g^{\mu\nu}(X) \quad (2.22)$$

where b is a constant to be determined with the requirement that

$$|g| \neq 0\ ) \quad (2.23)$$

These generalized structure functions can now also be written as

$$g^{\mu\nu}(X) = [D^{\mu}, X^{\nu}]/b \quad (2.24)$$

where $g^{\mu\nu}(X)$ are assumed to be analytic with $g_{\mu\nu}(X)$ defined by

$$g_{\mu\alpha}(X)\, g^{\alpha\nu}(X) = \delta_{\mu}^{\ \nu} \text{ in the X diagonal representation space.} \quad (2.25)$$

Then using (2.21) one gets

$$X^{\mu}(s+ds) - X^{\mu}(s) = dX^{\mu} = ds\, \eta_{\lambda}\, g^{\mu\lambda}(X) = ds\, \eta^{\mu}. \quad (2.26)$$

Then $g_{\mu\nu}(X)\, dX^{\mu} dX^{\nu} = ds^2\ g_{\mu\nu}(X)\, \eta^{\mu} \eta^{\nu} = ds^2$ since η^{μ} is a unit vector on this metric, thus (2.27)

$$ds^2 = g_{\mu\nu}(X)\, dX^{\mu} dX^{\nu} \text{ proving that } g_{\mu\nu}(X) \text{ is the metric for the space.} \quad (2.28)$$

One seeks transformations that will infinitesimally translate one in the X^{μ} space in order to study changes in the system. (In physics, as such changes are normally linked to the passage of time, then time itself must be one of the operators which we set to be the X^0 variable. But time is normally measured in seconds while space is measured in meters so we must convert our unit of time, second, to a unit of space, the meter, by using the invariant speed of light, c, and write X^0 as having the eigenvalue ct. Likewise, one could require all the X eigenvalues to be in seconds instead of meters.)

In the position representation one now has the representation for D as the differential operator:

$$\langle y|\ [D^{\mu}, X^{\nu}] = [(b\, g^{\mu\lambda}(y)\, (\partial/\partial y^{\lambda}) + A^{\mu}(y)\, ,\, y^{\nu}]\ \langle y| = [(\, b\, \partial^{\mu} + A^{\mu}(y)\,), y^{\nu}]\langle y| = b\, g^{\mu\nu}(y)\, \langle y| \quad (2.29)$$

Thus $\langle y|\ D^{\mu} = (\, b\, \partial^{\mu} + A^{\mu}(y)\,)\ \langle y|$ which allows the D commutator to represent derivatives: (2.30)

$$\langle y|\ D^{\mu}|\Psi\rangle = [b\, g^{\mu\upsilon}(y)\, ((\partial/\partial y^{\nu}) + A^{\mu}(y)\,)\, ,\, \Psi(y)] = b\, \partial^{\mu}\, \Psi(y) \text{ since } [A^{\mu}(y), \Psi(y)] = 0 \quad (2.31)$$

and where $\Psi(y) = \langle y|\Psi\rangle$ and $\partial^{\mu} = g^{\mu\upsilon}(y)\, (\partial/\partial y^{\nu})$ (2.32)

and $A^{\mu}(y)$ is a yet undetermined vector function of X^{ν}.

The Christoffel symbols are given by

$$\Gamma_{\gamma\alpha\beta} = (½)\, (\partial_{\beta}, g_{\gamma\alpha} + \partial_{\alpha}, g_{\gamma\beta} - \partial_{\gamma}, g_{\alpha\beta}\,) \quad (2.33)$$

and can be written in the position diagonal representation, in terms of the commutators of D with the metric as

$$\Gamma_{\gamma\alpha\beta} = (½)\, (1/b)\, (\, [D_{\beta}, g_{\gamma\alpha}] + [D_{\alpha}, g_{\gamma\beta}] - [\, D_{\gamma}, g_{\alpha\beta}\,]\ ). \quad (2.34)$$

Then using

$$g_{\alpha\beta}(X) = (1/b)\, [D_{\alpha}, X_{\beta}] \text{ one obtains} \quad (2.35)$$

$$\Gamma_{\gamma\alpha\beta} = (½)\, (b^{-2})\, (\, [D_{\beta}, [D_{\gamma}, X_{\alpha}]] + [D_{\alpha}, [D_{\gamma}, X_{\beta}]] - [\, D_{\gamma}, [D_{\alpha}, X_{\beta}]\,]\ ). \quad (2.36)$$

The Riemann tensor then becomes:

$$R_{\lambda\alpha\beta\gamma} = (1/b)\, (\, [D_{\beta}, \Gamma_{\lambda\alpha\gamma}\,] - [D_{\gamma}, \Gamma_{\lambda\alpha\beta}\,]\,) + (\Gamma_{\lambda\beta\sigma}\Gamma^{\sigma}_{\ \alpha\gamma} - \Gamma_{\lambda\gamma\sigma}\Gamma^{\sigma}_{\ \alpha\beta}\,) \quad (2.37)$$

where $\Gamma_{\gamma\alpha\beta}$ is to be inserted as the Christoffel symbols giving only commutators. One then defines the Ricci tensor from the Riemann tensor as

$$R_{\alpha\beta} = g^{\mu\nu}\, R_{\alpha\mu\beta\nu} = (1/b)\, [D^{\mu}, X^{\nu}]\, R_{\alpha\mu\beta\nu} \text{ and also defines} \quad (2.38)$$

$$R \;=\; g^{\alpha\beta} R_{\alpha\beta} \;=\; (1/b)\,[D^{\alpha}, X^{\beta}]\, R_{\alpha\beta}\,. \qquad (2.39)$$

It is well known that the ordinary derivative of a scalar function, $V_\mu = \partial\Lambda/\partial y^\mu$, in Riemann geometry will transform under arbitrary coordinate transformations as a covariant vector. But such a derivative of a vector function of the coordinates will not transform as a tensor. The covariant derivative with respect to y^ν of a contravariant vector A^μ is given by

$$A^{\mu}{}_{,\nu} = \partial A^\mu/\partial y^\nu + A^\sigma \Gamma^{\mu}{}_{\sigma\nu} \qquad (2.40)$$

and the covariant derivative of a covariant vector A_μ is given by

$$A_{\mu,\nu} = \partial A_\mu/\partial y^\nu - A_\sigma \Gamma^{\sigma}{}_{\mu\nu} \qquad (2.41)$$

where both $A^{\mu}{}_{,\nu}$ and $A_{\mu,\nu}$ transform as tensors with respect to the metric $g^{\alpha\beta}$.

One recalls for Riemannian geometry that there is a Christoffel symbol on the right-hand side for each index of the tensor being differentiated. In this algebraic framework one can write the covariant differentiation of a contravariant vector A^μ as:

$$A^{\mu}{}_{,\nu} = i\,[D_\nu, A^\mu] + (½)A^\sigma \left([D_\nu, [D^\mu, X_\sigma]] + [D_\sigma, [D^\mu, X_\nu]] - [D^\mu, [D_\sigma, X_\nu]] \right) \qquad (2.42)$$

Since, by definition, A is at most a function of the X operators. Thus, one can write both the regular derivative (first term) and complete it with the index contraction with the Christoffel symbol (second term). It is important to distinguish this covariant differentiation from the regular differentiation that occurs as a representation of the operator D^μ in the position representation. It follows that we can write the covariant derivative of any tensor in the same way but with a contraction of the Christoffel symbol with each of the tensor indices as is well known in Riemannian geometry. The angle between any two vectors is also defined in the customary way using only the symmetric part of $g_{\mu\nu}(X)$. One recalls that only the symmetric part of the metric is used in Riemannian geometry to determine distance and angle since it is contracted with a symmetric expression in equation (2.28) as $ds^2 = g_{\mu\nu}(X)\, dX^\mu dX^\nu$.

There is another transformation that is critical, and that is the infinitesimal gradual change (rotation) in each "plane" of two of the X variables such as in the $\mu\nu$ plane which can be generated by the generator given by,

$$L^{\mu\nu} = X^\mu D^\nu - X^\nu D^\nu \text{ with } M^{\mu\nu} = \exp(d\eta_{\mu\nu} L^{\mu\nu}) \qquad (2.43)$$

as the associated transformation which gives the operator for rotations in the $\mu\nu$ plane for a vector X thus smoothly forming a new linear combination of the X^μ and X^ν variables.

Both the translations generated by D and the rotations generated by L are essential operations that usually are time dependent and can be used with other considerations to formulate the dynamical changes in the X space of variables as well as all other non-scalar objects.

Although the primary objective here is to lay a more general Lie algebra foundation to merge general relativity with quantum theory, it should be noted that <u>the work above is purely mathematical and will apply to all other domains of Riemannian Geometry such as abstract mathematics and applications in other areas such as economics where the space is vast in representing the production (and prices) of the 500 to 1,000 input-output variables that describe the economy</u> in any country since those variables, although independent, form a space that is not Euclidian.

3. Application to Physics: The Extended Poincare Algebra & Invariants:

We have already done the necessary background work to frame GR in terms of the operator algebra. The Einstein equations:

$$R_{\alpha\beta} - ½\, g_{\alpha\beta}\, R + g_{\alpha\beta}\, \Lambda = (8\, \pi\, G/c^4)\, T_{\alpha\beta} \text{ can now be written as} \qquad (3.1)$$

$$R_{\alpha\beta} - (i\hbar\, [D_\alpha, X_\beta])\, (½\, R - \Lambda) = (8\, \pi\, G/c^4)\, T_{\alpha\beta} \qquad (3.2)$$

where $R_{\alpha\beta}$ and R are now given in terms of commutators as shown above while $T_{\alpha\beta}$ is the energy-momentum tensor as determined by the SM.

The discussion up to this point is purely mathematical and reframes RG in terms of a GLA. We now first look at the foundational observable operators in physics when gravitation is not present with QT, SR, and the SM when space-time is Euclidian. The commuting X^μ operators in the physics setting are operators for events in space-time where the X^0 operator has the eigenvalue ct (c = the speed of light and t is time on the reference frame clock while the X^i operators have the eigenvalues of spatial position, y^i, where i ranges over three or more spatial dimensions. The D^μ translation operators in the associated dimensions refer to the generalized momentum in the associated direction. The Poincare Lie Algebra (PA) is the fundamental symmetry algebra of physics consisting of the infinitesimal generators of the four (4) translations in space time, D^μ, (μ,ν = 0, 1, 2, 3), and three (3) rotations & three (3) Lorentz infinitesimal generators $L^{\mu\nu}$ giving the antisymmetric tensor operator, $M^{\mu\nu}$. All physical theories must be invariant under the Lie group generated by this Lie algebra and all entities that exist must be representations of the Poincare algebra as expressed in the indexing of creation and annihilation operators.

As quantum theory is founded upon the relationship between momentum and position operation as defined in the HA, with $[E/c, ct] = [D^0 , X^0] = i\hbar$, and $[D^i, X^j] = -i\hbar\, g^{ij}$, then a full Lie algebra of space-time observables must also include the commuting four-position operators X^μ to provide the foundations of quantum theory, as well as, $M^{\mu\nu}$, the generators of the Lorentz group in a flat space-time. This previously led us to the Extended Poincare Algebra [20,21] (EPA) by adding a four-vector position operator, X^μ whose components are to be considered as fundamental observables using a manifestly covariant form of the HA. We briefly review this EPA algebra[12] whose representations define free particles (fields) in a space-time with no curvature. Our GLA above must reduce to this algebra when there are no forces or gravitation present. As the X^μ generate translations in momentum, they do not generate symmetry transformations or represent conserved quantities but do provide the critical observables of space-time. We choose the Minkowski metric $g^{\mu\nu} = (+1, -1, -1, -1)$ and write the HA in the covariant form as $[D^\mu, X^\nu] = I\, g^{\mu\upsilon}$ where I is an operator that commutes with all elements and has the unique eigenvalue "iħ" with $D^0 = E/c$, $D^1 = D_x$ ..., $X^0 = ct$, $X^1 = x$ … where c is the speed of light, E is energy, and t is time. Here the "I" operator is needed to make the fifteen (15) fundamental observables in this EPD Lie algebra (X, D, M, I) close into a true Lie algebra with the structure constants as follows:

$$[I, D^\mu] = [I, X^\nu] = [I, M^{\mu\nu}] = 0 \qquad (3.3)$$

thus I commutes with all operators and has "iħ" as the only eigenvalue thus:

$$[D^\mu, X^\nu] = i\hbar\, g^{\mu\upsilon} \qquad (3.4)$$

which is the covariant Heisenberg Lie algebra – the foundation of quantum theory,

$$[D^\mu, D^\nu] = 0 \qquad (3.5)$$

which insures noninterference of energy momentum measurements in all four dimensions.

$$[X^\mu, X^\nu] = 0 \tag{3.6}$$

which insures noninterference of time and position measurements in all four dimensions.

$$[M^{\mu\nu}, D^\lambda] = i\hbar\, (g^{\lambda\nu}\, D^\mu \;-\; g^{\lambda\mu}\, D^\nu) \tag{3.7}$$

which guarantees that D^μ transforms as a vector under $M^{\mu\nu}$ and

$$[M^{\mu\nu}, X^\lambda] = i\hbar\, (g^{\lambda\nu}\, X^\mu \;-\; g^{\lambda\mu}\, X^\nu)$$

which guarantees that X^λ also transforms as a vector under $M^{\mu\nu}$ and thus the Lorentz group is

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\hbar\, (g^{\nu\rho}M^{\mu\sigma} + g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma} - g^{\mu\sigma}M^{\nu\rho}). \tag{3.8}$$

which guarantees that $M^{\rho\sigma}$ transforms as a tensor under the Lorentz group generated by $M^{\mu\nu}$.

The representations of the Lorentz algebra are well-known [8, 9] and are straight forward but the extension to include the four-momentum with the Poincare algebra makes the determination of the Poincare representations somewhat complicated. But with our extension of the Poincare algebra to include a four-position operator, X^μ, the representations are clearer. Because X^μ is now in the algebra, one can now define the orbital angular momentum four-tensor, operator $L^{\mu\nu}$ as:

$$L^{\mu\nu} = X^\mu\, D^\nu - X^\nu\, D^\mu \tag{3.9}$$

From this it can easily be shown that

$$[L^{\mu\nu}, D^\lambda] = i\hbar\, (g^{\lambda\nu}\, D^\mu \;-\; g^{\lambda\mu}\, D^\nu) \tag{3.10}$$

$$[L^{\mu\nu}, X^\lambda] = i\hbar\, (g^{\lambda\nu}\, X^\mu \;-\; g^{\lambda\mu}\, X^\nu) \tag{3.11}$$

$$[L^{\mu\nu}, L^{\rho\sigma}] = i\hbar\, (g^{\nu\rho}L^{\mu\sigma} + g^{\mu\sigma}L^{\nu\rho} - g^{\nu\rho}L^{\mu\sigma} - g^{\mu\sigma}L^{\nu\rho}). \tag{3.12}$$

One can then define an intrinsic spin four-tensor as:

$$S^{\mu\nu} = M^{\mu\nu} - L^{\mu\nu} \tag{3.13}$$

with the result that

$$[S^{\mu\nu}, D^\lambda] = 0\,; \quad [S^{\mu\nu}, X^\lambda] = 0\,; \quad [S^{\mu\nu}, L^{\rho\sigma}] = 0; \text{ and} \tag{3.14}$$

$$[S^{\mu\nu}, S^{\rho\sigma}] = i\hbar\, (g^{\nu\rho}S^{\mu\sigma} + g^{\mu\sigma}S^{\nu\rho} - g^{\nu\rho}S^{\mu\sigma} - g^{\mu\sigma}S^{\nu\rho})) \text{ which is the Lorentz Lie algebra.} \tag{3.15}$$

Thus one can separate this EPA algebra into the product of two Lie algebras, the nine parameter HA (consisting of X, P, I) and the six parameter homogeneous Lorentz algebra (consisting of the $S^{\mu\nu}$). Thus, one can write all representations as products of the representations of the two algebras. For the HA one can choose the position representation:

$$X^\mu |\, y > \; = y^\mu |\, y > \tag{3.16}$$

or the momentum representation

$$D^\mu |\, k > \; = k^\mu |\, k > \tag{3.17}$$

or equivalently diagonalize the mass and the sign of the energy and three momenta as

$$D^\mu\, D_\mu = m^2 c^2;\ \ \varepsilon(D^0),\, k^i, \text{ with eigenstates written as } |m, \varepsilon(D^0), k^i > \tag{3.18}$$

The transformations between the position diagonal and momentum diagonal basis vectors are given by the Fourier transform $<y|k> = (2\pi)^{-2} \exp((g_{\mu\nu}\, y^\mu\, k^\nu)/(i\hbar))$ as is well known.

All representations of the homogeneous Lorentz group have been found by Bergmann and by Gelfand, Naimark, and Shapiro [2,9] to be given by the two Casimir operators b_0 and b_1 defined as:

$$b_0^2 + b_1^2 - 1 = \tfrac{1}{2}\, g_{\mu\rho}\, g_{\nu\sigma}\, S^{\mu\nu}\, S^{\rho\sigma} \tag{3.19}$$

where $b_0 = 0, \frac{1}{2}, 1, 3/2, \ldots(|b_1|-1)$ and where b_1 is a complex number defined by

$$b_0 b_1 = -\tfrac{1}{4}\ \varepsilon_{\mu\nu\rho\sigma}\, S^{\mu\nu}\, S^{\rho\sigma} \tag{3.20}$$

with the rotation Casimir operator as S^2 which has the spectrum $s(s+1)$ with the total spin

$$s = b_0\,,\ b_0+1, \ldots, (|b_1| - 1) \tag{3.21}$$

and the z component of spin:

$$\sigma = -s, -s+1, \ldots. s-1, s \tag{3.22}$$

Thus the homogeneous Lorentz algebra representation can be written as $| b_0 , b_1 , s, \sigma >$ which joined with the Heisenberg algebra gives the full representation space as either

$|k^{\mu}, b_0 , b_1 , s, \sigma > = a^{+}_{k, b0, b1, s,\sigma} |0>$ for the momentum representation or (3.23)

$|y^{\mu}, b_0 , b_1 , s, \sigma > = a^{+}_{y, b0,b1,s,\sigma} |0>$ for the position representation. (3.24)

To obtain the effective extended Poincare algebra, one must generalize the equations above by allowing the effective momentum, D^{μ}, to contain both the gravitational metric and the vector bosons A as in equation (2.29) and likewise write: $L^{\mu\nu} = X^{\mu} D^{\nu} - X^{\nu} D^{\mu}$. (3.25)

Generally, one can argue that that which exists in nature, $|\Psi>$, must be a representation space of the algebra of operators X and D which in empty space have the commutation rules of the extended Poincare algebra of X^{μ}, D^{μ}, and $M^{\mu\nu}$. Those representations are known but they allow all spins and masses representing the invariants of that algebra. But that is not what we find in nature rather one finds a very restricted collection of masses and associated spins that closely adhere to the standard model of particle physics: U(1) x SU(2) x SU(3) with multiple seemingly arbitrary parameters. This "Standard Model" has evolved over the last few decades with incredible success, but we do not have a foundational theory for it and do not have a way to determine the associated masses, spins, and associated parameters. But such quantization of the associated Casimir operations of the XPM algebra when P is replaced by D^{μ}, $M^{\mu\nu}$ by $L^{\mu\nu} + S^{\mu\nu}$ where $L^{\mu\nu} = X^{\mu}D^{\nu} - X^{\nu}D^{\mu}$ could lead us to certain quantized states that would shed light on the basis of the standard model.

Considering the effective momentum operator, $D^{\mu} = i\hbar (g^{\mu\upsilon}(y) (\partial/\partial y^{\nu}) + \Gamma^{\mu}_{...}(y) + A^{\mu} (y))$ one can see a substantial change from the simple customary formula , $D^{\mu} = i\hbar (g^{\mu\upsilon} (\partial/\partial y^{\nu}) + A^{\mu} (y))$ because the metric $g^{\mu\upsilon}(y)$ has now become a function of the space time variables (eigenvalues y) $g^{\mu\upsilon}(y)$ and one now has the additional Christoffel symbol $\Gamma^{\mu}_{...}(y)$ needed to maintain full covariance in curved spaces. The vector fields can contain both the electroweak particles and the strong gluons. Invariance under the gauge transformations, $|\Psi'> = e^{\Lambda(x)}|\Psi>$ are thus altered by the presence of the gravitational field making the gauge transformations much more complex when one has electroweak and strong forces in very strong gravitational fields. One notes that this is a consequence of this approach and not an assumption.

Also, in a strong gravitational field near a star, such as a non-rotating white dwarf, one can treat the metric as constant using the Schwarzschild solution over a region that is small relative to the size of the star. The radial direction can be taken as the y^1 direction as the distance to the center of the star, with

$g_{00} = (1 - r_s/y^1)$ and $g_{11} = -1/(1 - r_s/y^1)$ (3.26)

where $r_s = 2GM/c^2$ with $g_{22} = g_{33} = -1$ (3.27)

and where G is the gravitational constant, M is the mass of the star, c is the speed of light, and y^1 is the distance to the center of the star, and r_s is the radius of the star, giving g(X) as the Schwarzschild solution. Equation (3.8) is still exactly the classical equation for the metric but recast in commutation relations (3.9). We therefore require that equation (3.9) be satisfied to determine the metric. However, there are certain results that follow that could potentially test our approach: The position and momentum operators are now to have the interpretational structure given by quantum theory with free particles as representations of the Extended Poincare Algebra. One essential new feature is that by virtue of the presence of a particle in a gravitational field such as near a star, the commutation rules with the rationalized Planks constant, ħ, are effectively modified by the metric in the radial (X^1) and time (X^0)

directions with the specific prediction that in a small region, with the Schwarzschild metric, one gets the altered uncertainty principles:

$$\Delta X_r \, \Delta D_r \geq (\hbar/2)(1/(1-r_s/r)) \text{ and } \Delta t \, \Delta E_r \geq (\hbar/2)\,(1-r_s/r) \tag{3.28}$$

where $r_s = 2GM/c^2$ and where r = the distance to the center of the spherical mass. This is because the generalized algebra effectively alters the value of Planks constant in both the X^1 and X^0 directions as well as the wave nature of particles in the altered local Fourier transform. This would in turn alter the creation rate of virtual pairs in the vacuum in a strong gravitational field. What is maintained is a more general form of the Heisenberg uncertainty principle obtained by multiplying (3.33) and (3.34) together to obtain:

$$\Delta t \, \Delta E_r \, \Delta X_r \, \Delta D_r \geq (\hbar/2)^2 \tag{3.29}$$

while the other two uncertainty relations remain unchanged. The metric would be quantized along with the vector fields in D. We are investigating whether this change in the uncertainty relations affect the virtual pair production and could lead to observable shifts in atomic spectra with hydrogen.

There is another especially critical item to be addressed in the gravitational theory concerning the critically important γ^μ matrices that form the Dirac equation $(\gamma^\mu D_\mu - m)\,|\Psi\rangle$ for spin ½ particles. For a Euclidian space-time we use the metric $g^{00} = g_{00} = 1$ and $g^{ii} = g_{ii} = -1$. With the Schwarzschild solution this metric becomes $g^{00} = g_{00} = (1 - r_s/y^1)$ and $g^{11} = g_{11} = -(1 - r_s/y^1)^{-1}$ where $r_s = 2GM/c^2$ with $g_{22} = g_{33} = -1$. The γ matrices can normally be derived from the requirement that the probability of finding the particle is constant over time because γ^μ gives the flow of the probability (conservation of probability or charge). That standard method is difficult to apply when gravity is expressed in the metric as with Einstein's equations because it cannot be assumed that the γ matrices have constant elements. The problem is that the flow of probability as given by the γ^μ vector, is altered by the non-Euclidian metric thus requiring $\gamma(X)^\mu$ to be dependent upon the position of the particle in space-time as it evolves over time and the metric cannot be solved except in a very limited number of cases. But there is another method that can be used and that is the fact that the space-time dependent $\gamma(X)$ matrices must reduce to $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = g^{\mu\nu}$ for the Euclidian metric so that we can require this equation to also be valid for the Schwarzschild metric. One can show by direct multiplication that the solution for the Schwarzschild metric is accomplished by changing γ^0 from (0,I ; I,0) to $\gamma^0(X) = (0, (1- r_s/y_1)^{1/2}); (1- r_s/y_1)^{1/2}, 0)$ and changing γ^1 to γ^1 from $(0, \sigma^1; \sigma^1, 0)$ to $\gamma^0(X) = (0, - \sigma^1(1- r_s/y_1)^{-1/2}; - \sigma^1(1- r_s/y_1)^{-1/2}, 0)$ where $r_s = 2\,GM/c^2$ and y_1 is the distance to the center of a non-rotating mass (star) and where γ^2 and γ^3 are unchanged. A similar change in γ^μ can be achieved for the Kerr metric solution of a rotating star. This alteration of the γ^μ matrices from constants to a necessary dependence upon the position operators X^m changes the Dirac equation in a gravitational field as well as other expressions that involve the $\gamma^\mu(X)$ such as the energy momentum tensor. One realizes the complexity of these results when one considers that the energy momentum tensor that determines the metric on the right-hand side of the Einstein equation now contains $\gamma^\mu(X)$ as well as the $\gamma^\mu(X)$ that is now present in the other side of the equation. Thus, we propose that the full Dirac equation in the X diagonal representation as:

$$\langle y|(\gamma^\mu(y)\, D_\mu - m)\,|\Psi\rangle = 0 = \langle y|\, \gamma^\mu(y)\, ((i\hbar\, \partial/\partial y^\nu) + \Gamma_{\mu\ldots}(y) + A_\mu\,(y)\,) - \mu)\,|\Psi\rangle \tag{3.30}$$

for spin ½ particles in gravitational, electroweak, and strong forces due to the standard model. The $\gamma^\mu(y)$ matrices are like a symmetrized "square root" of the metric, and this also alters the Lagrangian for spinor fields.

One can also consider another equation as a possible antisymmetric component of the metric $g_a{}^{\mu\nu}$. The angular momentum operator in the $\mu\nu$ plane which is given by

$$L^{\mu\nu} = X^{\mu}D^{\nu} - X^{\nu}D^{\mu} \quad (3.31)$$

or equivalently by the angular momentum density

$$m^{\mu\nu\rho} = X^{\mu} T^{\nu\rho} - X^{\nu} T^{\mu\rho}. \quad (3.32)$$

We can insert the expression for the energy momentum tensor in terms of the Einstein tensor G to get:

$$m_{\mu\nu\rho} = X_{\mu} T_{\nu\rho} - X_{\nu} T_{\mu\rho} = (X_{\mu} G_{\nu\rho} - X_{\nu} G_{\nu\rho})/(8 \pi G/c^4) \text{ or equivalently} \quad (3.33)$$

$$g_a{}^{\mu\nu} = \int d^4\sigma_{\rho}(X^{\mu} G^{\nu\rho} - X^{\nu} G^{\mu\rho}) = (8 \pi G/c^4) \int d^4\sigma_{\rho}\, m^{\mu\nu\rho} \quad (3.34)$$

which is an antisymmetric component of the metric which will not alter lengths or angles.

The representations of the homogeneous Lorentz group with D instead of P now give new results with the two Casimir operators C_0 and C_1 now defined as:

$$C_0 = g_{\mu\rho}\, g_{\nu\sigma}\, g_a{}^{\mu\nu}\, g_a{}^{\rho\sigma} \quad \text{and} \quad (3.35)$$

$$C_0 C_1 = \varepsilon_{\mu\nu\rho\sigma}\, g_a{}^{\mu\nu}\, g_a{}^{\rho\sigma} \quad (3.36)$$

We speculate that like Einstein’s equations, these equations determine other changes to the metric that result from the angular momentum density distribution. Since “dark matter” responds only to the gravitational force as expressed in the curvature of space time as detected in large scale galactic distributions of rotating mass distributions, then perhaps these equations could account for the effects in angular momentum in galaxies and similar large collections of matter, not due to the effective momentum. We are currently exploring this.

We are also studying (1) the gauge transformations as altered by the presence of the gravitational field in the standard model, (2) and the resulting antisymmetric component of the metric for a possible connection to dark matter effects. (3) The possible observable effects of the altered uncertainty principle as observable changes in the hydrogen spectra in a very strong gravitational field and other potential effects, and (4) Other ramifications of this approach such as the altered Dirac equation.

In keeping with relativistic quantum theory, particles (fields) are to be represented by creation and annihilation operators, indexed by representations of the (extended) Poincare algebra. It is notable that there is <u>essentially no freedom in this model,</u> except that the space time can be of any dimension, and that the system is subject to the ongoing discoveries and refinements of the SM with the metric being determined by the energy momentum tensor in keeping with Einstein’s equations.

4. Conclusions:

(a) The most important component of our work is that this theory is very tight and not dependent upon unknown models or assumptions but rather rests on one simple observation that a set of space-time observables that represent curvature as a Riemannian space (which we know to be the case from GR), must allow for the generator of translations D^{μ}, commuted with the observable X^{ν}, to depend upon the location in the space time structure and thus $[D^{\mu}, X^{\nu}] = b\, g^{\mu\nu}(X)$ where b must be given by $b = i\hbar$ in order to reduce to the standard Heisenberg equation. **(b)** The next most important result is that quantization relationship of the D and X observables in such an altered Heisenberg type Lie algebra is a consequence of generalizing the concept of a Lie algebra to have variable structure constants thus allowing the equations of GR to be expressed as commutation relations in this generalized Lie algebra. **(c)** The next most important conclusion is that the D operator in the position representation gives an exact framework for gravity (as $\mathbf{g}$ and $\boldsymbol{\Gamma}$) to be included in the same expression with the electroweak and strong fields (particles) and **(d)** integrating, without additional assumptions, with the SM. **(e)** The next important result is that the gauge transformations due to gravitation are generalized in a unique way by their expression with D. **(f)** Furthermore, by having an exact determination of D, one can immediately write the altered

Dirac equation as $(\gamma_\mu(X)\, D^\mu - m = 0)\, |\psi\rangle = 0$, Klein Gordan equation $(g_{\mu\nu}(X) D^\mu D^\nu - m^2)\, |\psi\rangle = 0$ etc**.** **(g)** There is also an exact prediction, (using the Schwarzschild metric) in terms of an altered uncertainty principle: $\Delta X_r\, \Delta D_r \geq (\hbar/2)(1/(1-r_s/r))$ and $\Delta t\, \Delta E_r \geq (\hbar/2)\,(1-r_s/r)$ and thus depend upon the strength of the gravitational field. As this is responsible for virtual pair production (vacuum polarization), it might be an observable effect for hydrogen spectra near the surface of a star or black hole. **(h)** The generator for rotations in the $\mu\nu$ plane is given by $L^{\mu\nu} = X^\mu D^\nu - X^\nu D^\nu$ with the altered effective momentum thus giving more complex commutation rules for the orbital angular momentum tensor in this system. **(i)** The antisymmetric tensor <u>$F^{\mu\nu} = [D^\mu , D^\nu]$ represents the associated forces of the standard model</u> with the (also rather complicated) presence of the Riemann metric. **(j)** The conserved effective momentum $D^\mu = i\hbar\, (g^{\mu\upsilon}(y)\, (\partial/\partial y^\nu) + \Gamma + A^\mu (y))$ both allows the intermediate bosons of the SM and combines it with the metric which must be quantized as a spin two field. This then integrates the metric (graviton) with the other boson fields explicitly for local gauge transformations which are now supported in this framework. <u>**(k)**</u> The conserved effective momentum $D^\mu = i\hbar\, (g^{\mu\upsilon}(y)\, (\partial/\partial y^\nu) + \Gamma + A^\mu (y))$ both allows the intermediate bosons of the SM and combines it with the metric which must be quantized as a spin two field. This then integrates the metric (graviton) with the other boson fields explicitly for local gauge transformations which are now supported in this framework. **(l)** We are also investigating whether the equations that add an antisymmetric component of the metric have possible implications such as for dark matter. **(m)** Finally, we have shown that it is possible to generalize a Lie algebra in an effective way by allowing the structure constants to become functions of an Abelian subalgebra (which is necessary to allow for noninterfering observations of the X variable (eigenvalues). **(n)** And finally, we have the Dirac equation for spin ½ particles in the position diagonal representation as $(g_{\mu\nu}(y)\, \gamma^\mu(y)\, D^\nu(y) - m)\, \langle y| = 0$ where D^μ is given above, and the Klein Gordan equation $(g_{\mu\nu}(y)\, D^\mu(y)\, D^\nu(y) - m^2)\, \langle y| = 0$ including gravitation and the standard model for the other forces.

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