Riemannian Geometry:
Framed as a Generalized Lie Algebra

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This paper reframes Riemannian geometry (RG) as a Generalized Lie algebra (GLA). We begin with an Abelian Lie algebra of n "position" operators, X, whose simultaneous eigenvalues, y, define a real n-dimensional space $\mathbb{R}(n)$. Then with n new operators defined as independent functions, $X'(X)$. we define contravariant and covariant tensors in terms of their eigenvalues. y, on a Hilbert space representation. We then define n additional operators, D,whose exponential map is, by definition, to translate X as defined by a noncommutative algebra of operators (observables) where the "structure constants" are shown to be the metric functions of the X operators to allow for spatial curvature which results in a noncommutativity among the D operators. The D operator then has a Hilbert space position-diagonal representation as a generalized differential operator plus a Christoffel symbol, $\Gamma^{\mu}(y)$, an arbitrary vector function $A^{\mu}(y)$, and the derivative of a scalar function $g^{\mu\nu}\frac{\partial\phi(y)}{\partial y^{\nu}}$. One can then express the Christoffel symbols, and the Riemann, Ricci, and other tensors as commutators in this representation, thereby framing RG as a GLA. This GLA provides a more general framework for RG to support an integration of general relativity, quantum theory and the standard model.
Keywords: Riemannian Geometry, Lie Algebra, Heisenberg Algebra

I. INTRODUCTION

Lie algebras, and the Lie groups which they generate, have played a central role in both mathematics and theoretical physics since their introduction by Sophius Lie in 1888 [1]. Both relativistic quantum theory (QT) and the gauge algebras of the phenomenological standard model (SM) of particles and their interactions are framed in terms of observables which form Lie algebras and are firmly established [2–5]. The Heisenberg Lie algebra (HA) among (generalized) momentum and position operators, $[D, X]$ gives the foundational structure of QT and has applications in mathematics in studies related to Fourier transforms and harmonic analysis [6–9],[10]. Likewise, in QT one has the Poincare symmetry Lie algebra (PA) of space-time observables whose representations define free particles. But the theory of gravitation as expressed in Einstein's general theory of relativity (GR), although also firmly established, is formulated in terms of a Riemannian geometry (RG) of a curved space-time where the metric is determined by nonlinear differential equations from the distribution of matter and energy [11, 12]. In GR there are no operators representing observables, and thus no commutation rules to define Lie algebras, and thus no representations of such algebras. The observables in GR are (a) the positions of events in space-time, and (b) the metric function of position in space-time (and its derivatives) which define the distance between events, and which define the curvature of space-time. Thus, QT and GR are expressed in totally different mathematical frameworks and their merger into a single theory has been a central problem in physics for over a century. However, the space-time events in QT are the eigenvalues of the space-time operators which are an essential part of the HA which also contains the Minkowski metric which defines the associated translated distance when space-time is not curved. If the associated space were curved, one would have a metric that was a function of the position in space-time. Such a generalized HA would no longer allow closure as a traditional Lie algebra but rather closure in the enveloping algebra of analytic functions of the basis elements of the Lie algebra. This led us to consider a Generalized Lie Algebra (GLA) generalizing the framework of a Lie algebra, with n space-time operators, X^{μ}, and n corresponding operators D^{μ}, which by definition are to execute infinitesimal translations in the associated representation space of the X^{μ} eigenvalues, y^{μ} ($\mu = 0, 1, n-1$). The X^{μ} are to form an Abelian algebra whose eigenvalues can represent a "space-time" manifold of four or a larger number of dimensions as the associated X eigenvalues are simultaneously measurable. But we allow the space-time to be curved, so the corresponding D^{μ} operators translate the X operators in a manner that depends upon the value of the X which alters their commutativity so the D^{μ} now interfere with each other. We found that this approach automatically generalized the HA "structure constants" to be proportional to the Riemann metric thus allowing the metric to be a function of the position operators, X[13]. This generalizes the concept of Lie algebra to allow for "structure constants" that are functions of the X operators in the algebra and thus are no longer constants except approximately in small neighborhoods. This paper first formally reframes RG [14] as a GLA including the HA. We show that the fundamental concepts in RG such as the coordinate transformations. contravariant and covariant tensors, Christoffel symbols, Riemann and Ricci tensors, and the Riemann covariant derivative can now all be expressed in terms of commutation relations among these fundamental operators. This framework is reminiscent of contractions of Lie algebras where the structure constants as functions are modified to vary smoothly among different algebras based upon certain external parameters [15–20] but not in the algebra itself as we propose. In a similar way, our algebra allows the structure constants to be dependent upon the X operators in the algebra so that RG is retrieved as a position-diagonal representation of the algebra as one moves over the Riemann manifold of X eigenvalues.

II. RIEMANNIAN GEOMETRY FRAMED AS A NONCOMMUTATIVE ALGEBRAIC GEOMETRY OF OBSERVABLES

We begin by defining a purely mathematical structure. Consider a set of n independent linear self-adjoint operators, X^μ, which form an Abelian Lie algebra of order n, where

$$[X^\mu, X^\nu] = 0 \text{ and where } \mu, \nu \in \{0, 1, 2, \dots, n-1\}. \quad (1)$$

It is assumed that the units of measurement are the same for the eigenvalues of all X variables. Consider a Hilbert Space of square integrable complex functions $|\Psi\rangle$ as a representation space for this algebra where a scalar product is used to normalize the vectors to unity, that is $\langle\Psi|\Psi\rangle = 1$. The simultaneous eigenvectors of the Abelian Lie algebra can be written as the outer product of the X^μ eigenvectors with the Dirac notation:

$$|y^0\rangle\,|y^1\rangle \dots |y^{n-1}\rangle = |y^0, y^1, \dots, y^{n-1}\rangle = |y\rangle \quad (2)$$

where the eigenvalues y^μ label the associated eigenvectors $|y\rangle$ of the X^μ operators where we use the notation

$$X^\mu\,|y\rangle = y^\mu\,|y\rangle \quad (3)$$

where the y^μ are real numbers defining the Hilbert manifold.

These independent real variables y^μ can be thought of as the coordinates (or basis vectors) of an n-dimensional space $\mathbb{R}_n$ since each set of values defines a point in $\mathbb{R}_n$. Let the eigenvalues be normalized to be orthonormal with the scalar product

$$\langle y_a|y_b\rangle = \delta(y_a^0 - y_b^0)\delta(y_a^1 - y_b^1)\dots\delta(y_a^{n-1} - y_b^{n-1}). \quad (4)$$

Let the decomposition of unity:

$$1 = \int \mathrm{d}y\,|y\rangle\,\langle y| \quad (5)$$

project the entire space onto the basis vectors $|y\rangle$ where $\langle y|$ is the dual vector to $|y\rangle$. A general vector in the representation (Hilbert) space of this Lie algebra can then be written as

$$|\Psi\rangle = \int \mathrm{d}y\,|y\rangle\,\langle y|\Psi\rangle = \int \mathrm{d}y\Psi(y)\,|y\rangle \quad (6)$$

where the function $\Psi(y)$ gives the "components" of the abstract vector $|\Psi\rangle$ on the basis vectors $|y\rangle$. Thus

$$\langle\Psi|\Psi\rangle = 1 = \int \mathrm{d}y\,\langle\Psi|y\rangle\,\langle y|\Psi\rangle = \int \mathrm{d}y\Psi^*(y)\Psi(y). \quad (7)$$

Now consider another set of n linear operators, X'^μ, which are independent analytic functions, $X'^\mu(X^\mu)$, of the X^μ operators also forming an Abelian Lie algebra on the same representation space for this algebra where it follows that:

$$[X'^\mu, X'^\nu] = 0. \quad (8)$$

Let the X'^μ have eigenvectors $|y'\rangle$ and eigenvalues y'^μ given by

$$X'^\mu\,|y'\rangle = y'^\mu\,|y'\rangle \quad (9)$$

where y'^μ are real numbers. The same orthonormality and decomposition of unity also obtain for the $|y'\rangle$ vectors which are also to be a complete basis for the space $|\Psi\rangle$. Then we can let the $X'^\mu(X^\nu)$ act to the left on the dual vector $\langle y'|$ and act to the right on the vector $|y\rangle$ as

$$\langle y'|X'^\mu|y\rangle = \langle y'|X'^\mu(X^\nu)|y\rangle \quad (10)$$

to give

$$y'^\mu\,\langle y'|y\rangle = y'^\mu(y)\,\langle y'|y\rangle\,. \quad (11)$$

Thus, the eigenvalues $y'^\mu = y'^\mu(y)$ give the transformation from the y coordinates to the y' coordinates if the Jacobian does not vanish $|\partial y'^\mu/\partial y^\nu| \neq 0$ which we require to be the case. Thus, the operator functions $X'^\mu(X^\nu)$ define a coordinate transformation in $\mathbb{R}_n$ between the eigenvalues (coordinates) y and the eigenvalues y' (transformed coordinates) that define $\mathbb{R}_n$. Then the set of n real variables y^μ and the alternative variables y'^μ both can be interpreted as specifying the coordinates of points in this n-dimensional real space $\mathbb{R}_n$ with coordinate transformations given by the functions

$$y'^\mu = y'^\mu(y). \quad (12)$$

It now follows that

$$\mathrm{d}y'^\mu = \frac{\partial y'^\mu}{\partial y^\nu}\mathrm{d}y^\nu \quad (13)$$

and any set of n functions $V^\mu(y)$ that transforms as the coordinates,

$$V'^\mu(y') = \left(\frac{\partial y'^\mu}{\partial y^\nu}\right) V^\nu(y) \quad (14)$$

is to be called a contravarient vector. The upper (contravarient) indices are normally taken as the variables that are normally measured while the lower (covarient)indices are obtained by lowering the index with the metric as shown below. We use the summation convention for repeated identical indices. The derivatives $\partial/\partial y^\nu$ transform as

$$\frac{\partial}{\partial y'^\nu} = \left(\frac{\partial y^\nu}{\partial y'_\mu}\right)\frac{\partial}{\partial y^\nu} \tag{15}$$

and any such vector $V^\mu(y)$ which transforms in this manner as

$$V'^\mu(y') = \left(\frac{\partial y^\nu}{\partial y'_\mu}\right)V_\nu(y) \tag{16}$$

is defined as a covarient vector. Upper indices are defined as contravariant indices while lower indices are covariant indices. Functions with multiple upper and lower indices that transform as the contravariant and covariant indices just shown are defined as tensors of the rank of the associated indices. One would like to have transformations that translate one in the $\mathbb{R}_n$ space of the operators X (and thus their eigenvalues y). We define a new additional set of n operators, D^μ, that translate a point an infinitesimal distance, $\mathrm{d}s$, in the $\mathbb{R}_n$ space respectively in each corresponding direction y^μ by using the transformation generated by the D^μ elements of the algebra via the exponential map with transformations:

$$G(\mathrm{d}s,\eta) = e^{\mathrm{d}s\eta_\mu D^\mu/b}. \tag{17}$$

In this transformation η_μ is defined to be a unit vector in the y space, b is an unspecified constant, and $\mathrm{d}s$ is defined to be the distance moved in the direction η_μ as defined below. Then

$$X'^\lambda = GX^\lambda G^{-1}. \tag{18}$$

By taking the translated distance $\mathrm{d}s$ to be infinitesimal, then one gets

$$\begin{aligned} X'^\lambda &= X^\lambda(s+\mathrm{d}s) \\ &= e^{\frac{\mathrm{d}s\eta_\mu D^\mu}{b}}X^\lambda(s)e^{-\frac{\mathrm{d}s\eta_\mu D^\mu}{b}} \\ &= \left(1+\frac{\mathrm{d}s\eta_\mu D^\mu}{b}\right)X^\lambda(s)\left(1-\frac{\mathrm{d}s\eta_\nu D^\nu}{b}\right) \\ &= X^\lambda(s)+\frac{\mathrm{d}s\eta_\mu[D^\mu,X^\lambda]}{b}+\mathcal{O}((\mathrm{d}s)^2)+\ldots \end{aligned} \tag{19}$$

Thus, the commutator $[D^\mu, X^\lambda]$, defines the way in which the transformations commute (interfere) with each other in executing the translations in keeping with the theory of Lie algebras and Lie groups although in general the D and X may not close as a standard Lie algebra. If the space is Euclidian (flat) then there is no dependence of the commutator upon location, and thus there is no interference among the D^μ. Then $[D^\mu, X^\lambda]$ can be normalized to $I\delta_\pm^{\mu\lambda}$ (since D^μ is defined to translate X^μ) thus:

$$[D^\mu, X^\lambda] = I\delta_\pm^{\mu\lambda} = b\delta_\pm^{\mu\lambda} \tag{20}$$

and the space is Euclidean (flat) where $\delta_\pm$ is the diagonal $n\times n$ matrix with ±1 on the diagonal with off-diagonal terms zero. The units of the constant b then are complementary to the units of the associated y eigenvalues since the product of the X and D eigenvalues must give the units of the constant b so that dimensional units balance for equation 20. This is the customary Heisenberg Lie algebra with structure constants $I\delta_\pm^\mu\lambda$ and with $[D^\mu, D^\lambda] = 0$ for $\mu \neq \lambda$. The additional operator, I, is to commute with all elements and by definition has a single eigenvalue b, and is needed to close the basis of the Lie algebra which now is of dimension $2n+1$. Thus, confirming that the distance is $\mathrm{d}s$:

$$\mathrm{d}X^\lambda(s) = \mathrm{d}s\eta_\mu b/b\delta_\pm^{\mu\lambda} = \mathrm{d}s\eta^\lambda + \mathcal{O}(ds). \tag{21}$$

We now wish to allow for curvature in the space $\mathbb{R}_n$ of the X eigenvalues. Thus the $[D, X]$ commutator is now allowed to be dependent upon the operators X and can vary from point to point in the non-Euclidian space. We define the functions $g^{\mu\nu}(X)$ as generalized structure functions (no longer constants) as:

$$[D^\mu, X^\nu] = bg^{\mu\nu}(X) \tag{22}$$

where b is a constant to be determined with the requirement that

$$|g| \neq 0. \tag{23}$$

It is necessary that the X operator subalgebra is Abelian because that allows the X operators to all be diagonalized simultaneously. Otherwise the space of the X eigenvalues would not allow the commutator to be well-defined at a point, and the space would be a set of fuzzy variables. These generalized structure functions can now also be written as

$$g^{\mu\nu}(X) = \frac{[D^\mu, X^\nu]}{b} \tag{24}$$

where $g^{\mu\nu}(X)$ are assumed to be analytic with $g_{\mu\nu}(X)$ defined by

$$g_{\mu\alpha}(X)g^{\alpha\nu} = \delta_\mu^\nu \tag{25}$$

in the X diagonal representation. Then using equation 21 one gets

$$X^\mu(s+\mathrm{d}s)-X^\mu(s) = \mathrm{d}X^\mu = \mathrm{d}s\eta_\lambda g^{\mu\lambda}(X) = \mathrm{d}s\eta^\mu. \tag{26}$$

Then

$$g_{\mu\nu}(X)\mathrm{d}X^\mu\mathrm{d}X^\nu = \mathrm{d}s^2 g_{\mu\nu}(X)\eta^\mu\eta^\nu = \mathrm{d}s^2 \tag{27}$$

since η^μ is a unit vector on this metric. Thus

$$ds^2 = g_{\mu\nu}(X)\mathrm{d}X^\mu \mathrm{d}X^\nu \tag{28}$$

proving that $g_{\mu\nu}(X)$ is the metric for the space.
One seeks transformations that will infinitesimally translate one in the X^μ space in order to study changes in the system. (In physics, as such changes are normally linked to the passage of time, then time itself must be one of the operators which we set to be the X^0 variable. But time is normally measured in seconds while space is measured in meters so we must convert our unit of time, second, to a unit of space, the meter, by using the invariant speed of light, c, and write X^0 as having the eigenvalue ct. Likewise, one could require all the X eigenvalues to be in seconds instead of meters.)
In the position representation one now has the representation for D as the differential operator:

$$\begin{aligned}\langle y|D^\mu|\Psi\rangle =&\Big(bg^{\mu\nu}(y)\frac{\partial}{\partial y^\nu} + \Gamma^\mu(y)\\ &+ A^\mu(y) + g^{\mu\nu}\frac{\partial\phi(y)}{\partial y^\nu}\Big)\Psi(y)\end{aligned} \tag{29}$$

and where

$$\Psi(y) = \langle y|\Psi\rangle \tag{30}$$

and which allows the D commutator to represent derivatives and where $A^\mu(y)$ and $\phi(y)$ are yet undetermined vector and scalar functions of X^ν and $\Gamma^\mu(y)$ is the required Christoffel symbol that is required when the derivative $g^{\mu\nu}(y)\frac{\partial}{\partial y^\nu}$ acts on a vector function. The Christoffel symbols are given by

$$\Gamma_{\gamma\alpha\beta} = \frac{1}{2}\left(\partial_\beta, g_{\gamma\alpha} + \partial_\alpha, g_{\gamma\beta} - \partial_\gamma, g_{\alpha\beta}\right) \tag{31}$$

and can be written in the position diagonal representation, in terms of the commutators of D with the metric as

$$\Gamma_{\gamma\alpha\beta} = \frac{1}{2}\frac{1}{b}[D_\beta, g_{\gamma\alpha}] + [D_\alpha, g_{\gamma\beta}] - [D_\gamma, g_{\alpha\beta}]. \tag{32}$$

Then using

$$g_{\alpha\beta}(X) = \frac{1}{b}[D_\alpha, X_\beta], \tag{33}$$

one obtains

$$\begin{aligned}\Gamma_{\gamma\alpha\beta} =&\frac{1}{2}b^{-2}\Big([D_\beta, [D_\gamma, X_\alpha]] +\\ &[D_\alpha, [D_\gamma, X_\beta]] - [D_\gamma, [D_\alpha, X_\beta]]\Big)\end{aligned} \tag{34}$$

The Riemann tensor then becomes

$$\begin{aligned}R_{\lambda\alpha\beta\gamma} =&\frac{1}{b}\left([D_\beta, \Gamma_{\lambda\alpha\gamma}] - [D_\gamma, \Gamma_{\lambda\alpha\beta}]\right) +\\ &+ \left(\Gamma_{\lambda\beta\sigma}\Gamma^\sigma_{\alpha\gamma} - \Gamma_{\lambda\gamma\sigma}\Gamma^\sigma_{\alpha\beta}\right)\end{aligned} \tag{35}$$

where $\Gamma_{\gamma\alpha\beta}$ is to be inserted as the Christoffel symbols giving only commutators. One then defines the Ricci tensor from the Riemann tensor as

$$R_{\alpha\beta} = g^{\mu\nu}R_{\alpha\mu\beta\nu} = \frac{1}{b}[D^\mu, X^\nu]R_{\alpha\mu\beta\nu} \tag{36}$$

and also defines

$$R = g^{\alpha\beta}R_{\alpha\beta} = \frac{1}{b}[D^\alpha, X^\beta]R_{\alpha\beta}. \tag{37}$$

It is well known that the ordinary derivative of a scalar function, $V_\mu = \partial\phi(y)/\partial y^\nu$, in Riemann geometry will transform under arbitrary coordinate transformations as a covariant vector. But such a derivative of a vector function of the coordinates will not transform as a tensor. The covariant derivative with respect to y^ν of a contravariant vector A^μ is given by

$$A^\mu_{,\nu} = \frac{\partial A^\mu}{\partial y^\nu} + A^\sigma\Gamma^\mu_{\sigma\nu} \tag{38}$$

and the covariant derivative of a covariant vector A_μ is given by

$$A_{\mu,\nu} = \frac{\partial A_\mu}{\partial y^\nu} + A_\sigma\Gamma^\sigma_{\mu\nu} \tag{39}$$

where both $A^\mu_{,\nu}$ and $A_{\mu,\nu}$ transform as tensors with respect to the metric $g^{\alpha\beta}$.
One recalls for Riemannian geometry that there is a Christoffel symbol on the right-hand side for each index of the tensor being differentiated. In this algebraic framework one can write the covariant differentiation of a contravariant vector A^μ as:

$$\begin{aligned}A^\mu_{,\nu} =&i[D_\nu, A^\mu] + \frac{1}{2}A^\sigma\Big([D_\nu, [D^\mu, X_\sigma]] +\\ &+ [D_\sigma, [D^\mu, X_\nu]] - [D^\mu, [D_\sigma, X_\nu]]\Big)\end{aligned} \tag{40}$$

Since, by definition, A is at most a function of the X operators. Thus, one can write both the regular derivative (first term) and complete it with the index contraction with the Christoffel symbol (second term). It is important to distinguish this covariant differentiation from the regular differentiation that occurs as a representation of the operator D^μ in the position representation. It follows that we can write the covariant derivative of any tensor in the same way but with a contraction of the Christoffel symbol with each of the tensor indices as is well known in Riemannian geometry. The angle between any two vectors is also defined in the customary way using only the symmetric part of $g_{\mu\nu}(X)$. One recalls that only the symmetric part of the metric is used in Riemannian geometry to determine distance and angle since it is contracted with a symmetric expression in equation 28 as $ds^2 = g_{\mu\nu}\mathrm{d}X^\mu \mathrm{d}X^\nu$.

There is another transformation that is critical, and that is the infinitesimal gradual change (rotation) in each "plane" of two of the X variables such as in the $\mu\nu$ plane which can be generated by the generator given by,

$$L^{\mu\nu} = X^{\mu} D^{\nu} - X^{\nu} D^{\mu} \tag{41}$$

with

$$M = e^{d\eta_{\mu\nu} L^{\mu\nu}} \tag{42}$$

as the associated transformation which gives the operator for rotations in the $\mu\nu$ plane for a vector X thus smoothly forming a new linear combination of the X^{μ} and X^{ν} variables.

Both the translations generated by D and the rotations generated by L are essential operations that usually are time dependent and can be used with other considerations to formulate the dynamical changes in the X space of variables as well as all other non-scalar objects.

III. CONCLUSION AND APPLICATIONS

Although the primary objective here is to lay a more general Lie algebra foundation to merge general relativity with quantum theory, it should be noted that the work above is purely mathematical and will apply to all other domains of Riemannian Geometry such as abstract mathematics and applications in other areas such as economics where the space is vast in representing the production (and prices) of the 500 to 1,000 input-output variables that describe the economy in any country since those variables, although independent, form a space that is not Euclidean. We have already done the necessary background work to frame GR in terms of the operator algebra. The Einstein equations:

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + g_{\alpha\beta}\Lambda = \left(\frac{8\pi G}{c^4}\right) T_{\alpha\beta} \tag{43}$$

can now be written as

$$R_{\alpha\beta} - (i\hbar[D_{\alpha}, X_{\beta}]) \left(\frac{1}{2}R - \Lambda\right) = \frac{8\pi G}{c^4} T_{\alpha\beta} \tag{44}$$

where $R_{\alpha\beta}$ and R are now given in terms of commutators as shown above while $T_{\alpha\beta}$ is the energy-momentum tensor as determined by the SM. The representation of quantum theory and the standard model within this GLA framework that includes gravity will be explored in a subsequent paper where we will also show that the effective momentum operator is

$$D^{\mu} = i\hbar \left(g^{\mu\nu}(y) \left(\frac{\partial}{\partial y^{\nu}}\right) + \Gamma^{\mu}(y) + A^{\mu}(y) + g^{\mu\nu} \frac{\partial \phi(y)}{\partial y^{\nu}} \right) \tag{45}$$

where the metric $g^{\mu\nu}(y)$ has now become a function of the space time variables and one now has the additional Christoffel symbol needed to maintain full covariance in curved spaces. In a strong gravitational field near a star, such as a non-rotating white dwarf, one can treat the metric as constant using the Schwarzschild solution over a region that is small relative to the size of the star. The radial direction can be taken as the y^1 direction as the distance to the center of the star, with

$$g_{00} = 1 - \frac{r_s}{y^1} \tag{46}$$

and

$$g_{11} = -\frac{1}{1 - r_s/y^1} \tag{47}$$

where $r_s = 2GM/c^2$ with $g_{22} = g_{33} = -1$ and where G is the gravitational constant, M is the mass of the star, c is the speed of light, y^1 is the distance to the center of the star, and r_s is the radius of the star, giving $g(X)$ as the Schwarzschild solution.
Thus one gets altered uncertainty principles:

$$\Delta X_r \Delta D_r \geq \frac{\hbar}{2}\left(\frac{1}{1 - r_s/r}\right) \tag{48}$$

and

$$\Delta t \Delta E \geq \frac{\hbar}{2}\left(1 - \frac{r_s}{r}\right) \tag{49}$$

where $r_s = 2GM/c^2$ and where r is the distance to the center of the spherical mass. What is maintained is

$$\Delta t \Delta E \Delta X_r \Delta D_r \geq \left(\frac{\hbar}{2}\right)^2 \tag{50}$$

while the other two uncertainty relations remain unchanged. We are investigating whether this change in the uncertainty relations affects the virtual pair production and could lead to observable shifts in atomic spectra with hydrogen. The space-time dependent $\gamma(X)$ matrices must reduce to $\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = g^{\mu\nu}$ for the Euclidean metric for the conservation of probability, so this equation must also be valid for the Schwarzschild metric. One can show by direct multiplication that the solution for the Schwarzschild metric is accomplished by changing:

$$\gamma^0 = \begin{bmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{bmatrix} \rightarrow \gamma^0(X) = \begin{bmatrix} 0 & \sqrt{1 - r_s/y_1} \\ \sqrt{1 - r_s/y_1} & 0 \end{bmatrix} \tag{51}$$

$$\gamma^1 = \begin{bmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{bmatrix} \rightarrow \gamma^1(X) = \begin{bmatrix} 0 & -\frac{\sigma^1}{\sqrt{1-r_s/y_1}} \\ -\frac{\sigma^1}{\sqrt{1-r_s/y_1}} & 0 \end{bmatrix} \tag{52}$$

where $r_s = 2GM/c^2$ and y_1 is the distance to the center of a non-rotating mass (star) and where γ^2 and γ^3

are unchanged. Thus, the full Dirac equation in the X diagonal representation is:

$$0 = \langle y|\gamma^\mu(y) D_\mu - m|\Psi\rangle \tag{53}$$

$$= \left(\gamma^\mu(y)\left[i\hbar\frac{\partial}{\partial y^\mu} + \Gamma_\mu(y) + A_\mu(y) + \frac{\partial}{\partial y^\mu}\right] - m\right)\langle y|\Psi\rangle\,. \tag{54}$$

Free particles would be represented by creation and annihilation operators indexed with the XPM algebra representations [21],[22].

[1] S. Lie, *Theorie der Transformations-gruppen I, II, III* (Unter Mitwirkung von F. Engel. Leipzig. 1893).

[2] S. Weinberg, *The Quantum Theory of Fields I, II, III* (Cambridge University Press, 1995).

[3] S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Row, Peterson, and Company, 1989).

[4] J. Donoghue, E. Golowich, and B. Holstein, *Dynamics of the Standard Model* (Cambridge University Press, 1994).

[5] M. D. Schwartz, *Quantum Field Theory and the Standard Mode* (Cambridge University Press, 2014).

[6] P. Woit, Topics in representation theory: The heisenberg algebra (2023).

[7] E. Binz and S. Pods, *Geometry of Heisenberg Groups* (American Mathematical Society, 2008).

[8] I. M. Gel'fand, R. Minlos, and Z. Y. Shapiro, *Representations of the Rotation and Lorentz Groups and Their Applications* (The Macmillan Co New York, 1963).

[9] B. C. Hall, *Graduate Texts in Mathematics: Lie Groups, Lie Algebras, and Representations: An Elementary Introduction* (Springer, 2015).

[10] R. Howe, On the role of the heisenberg group in harmonic analysis, Bull. Amer. Math. Soc. 3 (2): 821–843 (1980).

[11] S. Weinberg, *Gravitation and Cosmology* (John Wiley, 1972).

[12] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W. H. Freeman, 1973).

[13] J. E. Johnson, An integration of general relativity with quantum theory and the standard model (2016), arXiv:1606.00701 [physics.gen-ph].

[14] L. P. Eisenhart, *Riemannian Geometry* (Princeton University Press, 1997).

[15] Y. S. Kim, Group contractions: Inonu, wigner, and einstein (1996), arXiv:hep-th/9610183.

[16] A. H. Dooley and J. W. Rice, "on contractions of semisimple lie groups, Transactions of the American Mathematical Society. 289 (1): 185–202 (1985).

[17] R. Gilmore, *Dover Books on Mathematics: Lie Groups, Lie Algebras, and Some of Their Applications* (Dover Publications, 2006).

[18] E. Inonu and E. P. Wigner, On the contraction of groups and their representations, Proc. Natl. Acad. Sci. 39 (6): 510–24. (1953).

[19] E. J. Saletan, Contraction of lie groups, Journal of Mathematical Physics. 2 (1): 1 (1961).

[20] I. E. Segal, A class of operator algebras which are determined by groups, Duke Mathematical Journal. 18: 221 (1951).

[21] J. E. Johnson, Position operators and proper time in relativistic quantum mechanics, Phys. Rev Vol 181, 1755-1764 (1969).

[22] J. E. Johnson, Proper-time quantum mechanics ii, Phys. Rev. D, Vol 3, No. 8, 1735-1747 (1971).