PHYSICAL REVIEW D VOLUME 10, NUMBER 8 15 OCTOBER 1974

Exact diagonalization of the Dirac Hamiltonian in an external field

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(Received 13 December 1973; revised manuscript received 1 July 1974)

The Hamiltonian of a Dirac particle in an arbitrary electromagnetic field is exactly diagonalized by a unitary transformation generalizing previous work which was restricted to time-dependent fields. A very simple form is found for the covariant Heisenberg equations which manifestly exhibits the classical correspondence. These results are obtained in a manifestly covariant form using a previously proposed proper-time quantum mechanics with subsequent specialization to a mass eigenstate resulting in the usual theory. The simple theorem used for this diagonalization is also applied to other transformations for helicity and the free-particle Hamiltonian. The source of difficulty in obtaining these results without an intermediate use of proper-time theory is shown.

I. INTRODUCTION

Several interpretational aspects of the free Dirac equation were clarified in the classic paper by Foldy and Wouthuysen[1] in which a unitary transformation was found which diagonalized the Dirac Hamiltonian with respect to positive and negative energies. The application of this transformation to the basic operators of position, momenta, orbital angular momenta, and spin exhibited a separation in the new representation into classical and nonclassical portions. The classical terms obeyed Heisenberg equations formally resembling the equations of classical mechanics, while the nonclassical terms exhibited a rapid oscillatory motion about the classical values *(zitterbewegung)*. When electromagnetic interactions were included, the transformation could not be obtained in closed form. Thus the classical separation could not be effected and the Heisenberg equations were not studied. Furthermore, the general approach was noncovariant. Subsequent work by Eriksen[2] has shown a closed form for the transformation when the electromagnetic field is time-independent and is free of a scalar potential. Chakrabarti[3] has studied a covariant diagonalization, but dealt only with free particles. A general review of these and associated problems can be found in the work of de Vries.[4]

This paper addresses three problems: First, is there a manifestly covariant generalization of the Foldy-Wouthuysen transformation? Second, can this procedure be extended covariantly to include arbitrary electromagnetic interactions in closed form? Third, can a covariant form of the Heisenberg equations be found which explicitly shows the classical form even with an interaction present? An affirmative answer to these questions can be given in the context of a proper-time quantum mechanics which as been previously proposed by one of the authors.[5] Although we utilize the proper-time approach to maintain covariance, the results can be immediately specialized to the usual theory by using mass eigenstates.

We find that the covariance appears mandatory for the diagonalization in arbitrary fields. If one uses the noncovariant Hamiltonian $P^0=\beta m+\vec{\alpha}\cdot\vec{p}$ and performs the replacement $P^\mu\to P^\mu-eA^\mu$ one encounters the difficulty pointed out by Sucher[6] that the resulting square-root Klein-Gordon equation does not admit the Lorentz group. Furthermore, it is well known that a separation of positive- and negative-energy states (P^0 diagonalization) is not possible with time-dependent fields and scalar potentials. We are able to bypass these problems by diagonalizing the proper-time Hamiltonian $H=\gamma\cdot(P-eA)$, which is a Lorentz invariant for arbitrary fields. It should be mentioned that the proper-time theory is more general in that states do not have to be mass eigenstates. After the calculations are performed, one may restrict states to mass eigenstates, thus eliminating the proper-time variable and retrieving the usual theory. For a proper study of the Heisenberg equations we need not only the manifest covariance but also a well-defined four-position operator with a proper-time dynamics.

In the next section we will outline the main features of the covariant algebraic proper-time formulation of quantum mechanics which one of us has previously proposed by extending the Poincaré algebra to include a covariant position operator. This larger mathematical framework enables us to maintain covariance at each stage of the calculation and to interpret the results from a more group-theoretic point of view. In Sec. III we point out a simple useful theorem and exemplify its use in diagonalizing helicity. In Sec. IV we diagonalize the free-particle Hamiltonian generalizing Foldy's and Wouthuysen's results to covariant form. In Sec. V we diagonalize the Hamiltonian for a particle in an external field, and in Sec. VI we obtain and discuss equations of motion.

II. PROPER-TIME QUANTUM MECHANICS

In two previous papers[5] one of us has proposed an extension of the Poincaré algebra $(P^\mu, M^{\mu\nu})$ to a larger algebra $(X^\mu, P^\mu, M^{\mu\nu})$ to accommodate a covariant position operator X^μ. The algebra was defined by the commutation rules

$$[P^\mu, X^\nu]=ig^{\mu\nu}, \tag{1a}$$

$$[X^\mu, X^\nu]=0, \tag{1b}$$

$$[P^\mu, P^\nu]=0, \tag{1c}$$

$$[M^{\mu\nu}, X^\lambda]=i(g^{\lambda\nu}X^\mu-g^{\lambda\mu}X^\nu), \tag{1d}$$

$$[M^{\mu\nu}, P^\lambda]=i(g^{\lambda\nu}P^\mu-g^{\lambda\mu}P^\nu), \tag{1e}$$

$$[M^{\mu\nu}, M^{\rho\sigma}]=-i(g^{\mu\rho}M^{\nu\sigma}+g^{\nu\sigma}M^{\mu\rho}-g^{\nu\rho}M^{\mu\sigma}-g^{\mu\sigma}M^{\nu\rho}), \tag{1f}$$

where (1c), (1e), and (1f) define the Poincaré algebra, and where the position operator is required to be a four-vector (1d) with mutually commuting components (1b) which is translated by the physical four-momenta (1a).

Taken as the algebra of one-particle observables, it follows that the allowable states of a particle lie in the representation space of the algebra. All representations were found by noting that with the orbital angular momentum defined by

$$L^{\mu\nu}=X^\mu P^\nu-X^\nu P^\mu, \tag{2a}$$

and the intrinsic spin defined by

$$S^{\mu\nu}=M^{\mu\nu}-L^{\mu\nu}, \tag{2b}$$

it followed that the representations of the algebra are equivalent to the representations of the direct product of the von Neumann algebra X^μ, P^ν and the homogeneous Lorentz algebra $S^{\mu\nu}$. As these representations are known, one need only take the direct product of the two representation spaces, e.g. $|k^\mu\rangle$ or $|y^\mu\rangle$ as an eigenstate of P^μ or X^μ with $|b_0, b_1, s, \sigma\rangle$ as a representation of $S^{\mu\nu}$, where

$$b_0{}^2+b_1{}^2-1=\tfrac{1}{2}S_{\mu\nu}S^{\mu\nu}, \tag{3a}$$

$$b_0 b_1=\frac{i}{8}\epsilon_{\mu\nu\rho\sigma}S^{\mu\nu}S^{\rho\sigma} \tag{3b}$$

define the Casimir operators b_0 and b_1, while s is the eigenvalue of spin and σ is the third component. The dynamical development of any operator Q is given in the Heisenberg picture by

$$Q(\tau)=e^{-i\tau H}Qe^{i\tau H}, \tag{4}$$

where τ is the proper time and H is an invariant Hamiltonian formed from the enveloping algebra (i.e., function of X^μ, P^μ, and $M^{\mu\nu}$) and possibly other operators supported by their representation space. The proper time, τ, is a c-number scalar which parameterizes the dynamics.

The unique spin-$\frac{1}{2}$ representations (Dirac theory) are specified by eigenvalues of the Casimir operators $b_0=\frac{1}{2}$, $b_1=\pm\frac{3}{2}$; thus we abbreviate

$$|b_0, b_1, S, \sigma\rangle\to|\epsilon(b_1), \sigma\rangle, \tag{5}$$

where the sign of b_1 corresponds to the eigenvalue of $i\gamma_5$. Thus a complete basis for Dirac theory with four-momentum diagonal is $|k^\mu, \epsilon(b_1), \sigma\rangle$ and with four-position diagonal is $|y^\mu, \epsilon(b_1), \sigma\rangle$.

This representation space is much larger than the one used in the usual Dirac theory in the following ways. Rewriting the four momenta k^μ in terms of $m=(K_\mu k^\mu)^{1/2}$, $\epsilon(k^0)$, $\vec{k}$ the state is given by $|m, \vec{k}, \epsilon(k^0), \epsilon(i\gamma_5), \epsilon(\sigma)\rangle$. As all four momenta are mathematically possible, all real and imaginary m are included. Thus it is a physical requirement that one use a range of m which is physically acceptable and so m is real and positive. In the usual Dirac theory a single mass eigenstate is chosen. There is a second respect in which the space is larger as all three signs are independently positive or negative giving an eightcomponent space. By making the physical requirement that only those states are allowable which are positive eigenstates of the operator $\gamma\cdot P$, one gets states which satisfy the Dirac equation. This separation into $\gamma\cdot P>0$ and $\gamma\cdot P<0$ subspaces is invariant under space and time inversions and particle conjugation. From the point of view of group theory, we are decomposing a representation of the X, P, M algebra into irreducible Poincaré representations using the invariants m, $\epsilon(P^0)$, and $\gamma\cdot P$. Thus the usual Dirac equation can be viewed as two simultaneous requirements: first, that the physical state is a mass eigenstate $m|\psi\rangle = m_0|\psi\rangle$, and second, that it is a member of the $\gamma\cdot P>0$ subspace; thus $\gamma\cdot P|\psi\rangle = m|\psi\rangle$. By dealing with the larger space and the proper-time dynamics we can obtain covariant results which later can be specialized to that subspace of the Dirac theory.

III. THEOREM AND SIMPLE EXAMPLES

This article will heavily rely upon a very simple theorem which is especially useful in Dirac theory and which is suggested by the work of Foldy and Wouthuysen: Let A and B be operators such that $A^2 = B^2 = 1$ and which anticommute, $[A, B]_+ = 0$, and which are Hermitian, $A = A^\dagger$ and $B = B^\dagger$ (on the γ^0 Dirac metric). Then the unitary transformation $U = e^{\theta AB}$ is by definition

$$e^{\theta AB} = 1 + \theta AB - \frac{1}{2!}\theta^2 - \frac{1}{3!}\theta^3 AB + \cdots \tag{6}$$

or

$$e^{\theta AB} = \cos\theta + AB\sin\theta .$$

Also it will be noted that

$$e^{\theta AB} B e^{-\theta AB} = e^{2\theta AB} B , \tag{7}$$

which proves the theorem that

$$e^{\theta AB} B e^{-\theta AB} = B\cos 2\theta + A\sin 2\theta . \tag{8}$$

Thus the unitary transformation $e^{\theta AB}$ rotates both A and B in the space of the two. By choosing $\theta = \pi/4$ we get $A \to -B$ and $B \to A$. The utility of this transformation is that it takes one from the basis where B is diagonal to the basis where A is diagonal. In particular if $B|\beta\rangle = \beta|\beta\rangle$ then $e^{\theta AB}|\beta\rangle = |\beta\rangle'$ such that $A|\beta\rangle' = \beta|\beta\rangle'$. This may be seen from

$$e^{\theta AB} B e^{-\theta AB} e^{\theta AB}|\beta\rangle = A e^{\theta AB}|\beta\rangle = \beta e^{\theta AB}|\beta\rangle . \tag{9}$$

We first exemplify the theorem by diagonalizing the helicity. Define $\omega = 2(\vec{S}\cdot\vec{P}/|\vec{P}|)$ and $\sigma_3 = 2S_3$ so that both operators have the spectrum ± 1. Then we would like to find the unitary transformation $U\sigma U^{-1} = \omega$, where U is in the form $e^{i\theta H}$. Here we can apply the theorem with

$$A = (P_1\sigma_1 + P_2\sigma_2)/(P_1^2 + P_2^2)^{1/2} \tag{10a}$$

and

$$B = \sigma_3 , \tag{10b}$$

and rotating B in the AB space,

$$e^{\theta AB} B e^{-\theta AB} = B\cos 2\theta + A\sin 2\theta . \tag{11}$$

We require that this result be equal to ω. One easily verifies that $[A, B]_+ = 0$ and $A^2 = 1 = B^2$, and that A and B are Hermitian. Thus we get

$$\sigma_3\cos 2\theta + [(P_1\sigma_1 + P_2\sigma_2)/(P_1^2 + P_2^2)^{1/2}]\sin 2\theta = \frac{\vec{\sigma}\cdot\vec{P}}{|P|} . \tag{12}$$

Thus

$$\cos 2\theta = \frac{P_3}{P} \tag{13a}$$

and

$$\sin 2\theta = \frac{(P_1^2 + P_2^2)^{1/2}}{P} , \tag{13b}$$

which are consistent and which determine θ to be $\frac{1}{2}\cos^{-1}(P_3/P)$. Also

$$\begin{aligned} e^{\theta AB} &= \cos\theta + AB\sin\theta \\ &= \frac{1}{\sqrt{2}}\left\{(1 + P_3/P)^{1/2} + \left(\frac{P_1\sigma_1 + P_2\sigma_2}{(P_1^2 + P_2^2)^{1/2}}\right)\sigma_3(1 - P_3/P)^{1/2}\right\} \end{aligned} \tag{14}$$

which gives the explicit form of the transformation.

As another application we consider the transformation $i\gamma_5$ to γ^0, both of which satisfy the necessary criterion. The transformation is $U = e^{\theta\gamma^0 i\gamma_5}$ and thus

$$e^{\theta\gamma^0 i\gamma_5}(i\gamma_5)e^{-\theta\gamma^0 i\gamma_5} = i\gamma_5\cos 2\theta + \gamma^0\sin 2\theta , \tag{15}$$

in which we set $\theta = \pi/4$ to get $Ui\gamma_5 U^\dagger = \gamma^0$. U may be written

$$e^{\theta\gamma^0 i\gamma_5} = \frac{1}{\sqrt{2}}(1 + \gamma^0 i\gamma_5) . \tag{16}$$

IV. FREE PARTICLE

The proper-time dynamics for the Dirac particle in the Heisenberg picture for any operator Q is $Q(\tau)=e^{-i\tau H}Qe^{i\tau H}$ or $dQ/d\tau=i[Q,H]$, where $H=\gamma_\mu P^\mu$. This gives the Heisenberg equations

$$\dot{X}^\mu=\gamma^\mu, \tag{17a}$$

$$\dot{P}^\mu=0, \tag{17b}$$

$$\dot{\gamma}^\mu=4S^{\mu\nu}P_\nu, \tag{17c}$$

$$\dot{\gamma}^\mu=2i\gamma^\mu H-2iP^\mu, \tag{17d}$$

$$\dot{S}^{\mu\nu}=-\gamma^\mu P^\nu+\gamma^\nu P^\mu, \tag{17e}$$

$$\dot{L}^{\mu\nu}=\gamma^\mu P^\nu-\gamma^\nu P^\mu. \tag{17f}$$

The $\dot{\gamma}^\mu$ equation is solvable:

$$\frac{d}{d\tau}\left(\gamma^\mu-\frac{P^\mu}{H}\right)=2i\left(\gamma^\mu-\frac{P^\mu}{H}\right)H, \tag{18}$$

thus giving

$$\gamma^\mu(\tau)=\frac{P^\mu}{H}+\left(\gamma^\mu(0)-\frac{P^\mu}{H}\right)e^{2i\tau H}, \tag{19}$$

clearly exhibiting the classical four-velocity and *zitterbewegung* parts. The operator H^{-1} means $(1/m^2)\gamma\cdot P$, where $m^2=P_\mu P^\mu$. Inserting $\gamma^\mu(\tau)$ into the equation one easily finds the solution

$$X^\mu(\tau)=X^\mu(0)+\frac{P^\mu}{H}\tau+\left(\gamma^\mu(0)-\frac{P^\mu}{H}\right)(e^{2iH}-1)\frac{1}{2iH}, \tag{20}$$

which again shows the separate classical motion in covariant form. This may also be written

$$X^\mu(\tau)=X^\mu(0)+\frac{P^\mu}{H}\tau+[\gamma^\mu(\tau)-\gamma^\mu(0)]\frac{1}{2iH}. \tag{21}$$

This solution along with $P^\mu(\tau)=P^\mu(0)$ solves the $\dot{L}^{\mu\nu}$ equation as

$$L^{\mu\nu}(\tau)=L^{\mu\nu}(0)+T^{\mu\nu}(\tau), \tag{22}$$

where

$$T^{\mu\nu}(\tau)=[\gamma^\mu(0)P^\nu-\gamma^\nu(0)P^\mu](e^{2iH}-1)\frac{1}{2iH} \tag{23}$$

and

$$S^{\mu\nu}(\tau)=S^{\mu\nu}(0)-T^{\mu\nu}(\tau). \tag{24}$$

Thus one sees that $M^{\mu\nu}(\tau)=L^{\mu\nu}(\tau)+S^{\mu\nu}(\tau)=M^{\mu\nu}(0)$, so that the total angular momentum is conserved. One notices that only the "classical" portions of $L^{\mu\nu}$ and $S^{\mu\nu}$ are separately conserved.

In view of the preceding equations one could define new "classical-like" operators by subtracting out the *zitterbewegung* operators from both sides of the equation to get "mean" operators X_m, L_m, and S_m, the latter two of which are separately conserved. But this method would not be readily extendable to an interacting particle where the exact solutions could not be found.

Foldy and Wouthuysen approached this problem noncovariantly by seeking a transformation which diagonalized the Hamiltonian P^0 and led to mean 3-vector operators X_m^i which exhibited classical behavior. We can accomplish this covariantly by diagonalizing the Hamiltonian $H=\gamma\cdot P$ and using the above theorem to first transform the initial state, $|k^\mu, i\gamma_5, \sigma\rangle$ to $|k^\mu, i\gamma_5, \omega^0\rangle$ by the helicity transformation of the last section. We then diagonalize $\gamma\cdot P$ using the theorem with $A=i\gamma_5$ and $B=\gamma\cdot P/m$ giving a state $|k^\mu, \epsilon(\gamma\cdot P), \omega_\epsilon^0\rangle$. The operator B is defined on $\gamma\cdot P$ eigenstate and thus as an integral operator on configuration space. Thus from the theorem

$$\begin{aligned} UHU^\dagger &= e^{+\theta i\gamma_5(\gamma\cdot P/m)}\gamma\cdot Pe^{-\theta i\gamma_5(\gamma\cdot P/m)} \\ &= \gamma\cdot P\cos 2\theta+i\gamma_5 m\sin 2\theta, \end{aligned} \tag{25}$$

in which we choose $\theta=\pi/4$ to get

$$UHU^\dagger=i\gamma_5 m, \tag{26}$$

with

$$U=\frac{1}{\sqrt{2}}\left(1+i\gamma_5\frac{\gamma\cdot P}{m}\right). \tag{27}$$

This transformation is the covariant generalization of the Foldy-Wouthuysen transformation. We can ask for the form which the basis of the algebra takes under the transformation U and one finds

$$P'^\mu=P^\mu, \tag{28a}$$

$$\gamma'^\mu=\gamma^\mu+\left(i\gamma_5-\frac{\gamma\cdot P}{m}\right)\frac{P^\mu}{m}, \tag{28b}$$

$$i\gamma_5'=-\frac{\gamma\cdot P}{m}, \tag{28c}$$

$$X'^\mu=X^\mu+\frac{i}{2}\left(i\gamma_5-\frac{\gamma\cdot P}{m}\right)\left(\frac{\gamma^\mu}{m}-\gamma\cdot P\frac{P^\mu}{m^3}\right), \tag{28d}$$

$$L'^{\mu\nu}=X'^\mu P^\nu-X'^\nu P^\mu, \tag{28e}$$

$$S'^{\mu\nu}=\tfrac{1}{4}i(\gamma'^\mu\gamma'^\nu-\gamma'^\nu\gamma'^\mu), \tag{28f}$$

$$(\gamma\cdot P)'=i\gamma_5 m. \tag{28g}$$

Since the transformation $e^{(\pi i/4)\gamma_5(\gamma\cdot P/m)}$ was unitary, all commutation rules are the same for the transformed algebra and in particular for their time development, i.e., the Heisenberg equations of motion. However, as the Hamiltonian is in diagonal form, one sees that the covariant position operator splits naturally into two parts, the first of which obeys classical-like equations and is the covariant generalization of the mean position operator of Foldy. One sees that it is defined in the old representation as $X_m^\mu\equiv U^\dagger X^\mu U$

so that $X_m'^{\mu} = UU^\dagger X^\mu UU^\dagger = X^\mu$ in the new representation. It is straightforward to check that the orbital angular momentum tensor defined with X_m^μ, i.e., $L_m^{\mu\nu} \equiv X_m^\mu P^\nu - X_m^\nu P^\mu$, is a conserved quantity. Consequently, the mean tensor $S_m^{\mu\nu} \equiv M^{\mu\nu} - L_m^{\mu\nu}$ is conserved. The Heisenberg equations in the new representation are

$$\dot{P}'^{\mu} = 0, \tag{29a}$$

$$\dot{X}'^{\mu} = i\gamma_5 \frac{P^\mu}{m} + \gamma^\mu - \frac{\gamma\cdot P}{m}\frac{P^\mu}{m}, \tag{29b}$$

$$\dot{L}'^{\mu\nu} = \gamma^\mu P^\nu - \gamma^\nu P^\mu \tag{29c}$$

$$\dot{\gamma}'^{\mu} = 2\gamma_5\left(\gamma^\mu m - \gamma\cdot P\frac{P^\mu}{m}\right), \tag{29d}$$

$$\dot{S}'^{\mu\nu} = -\gamma^\mu P^\nu + \gamma^\nu P^\mu . \tag{29e}$$

V. PARTICLE IN AN EXTERNAL FIELD

For the case of a particle interacting with an external field, P^μ is replaced by $P^\mu - eA^\mu(x) \equiv D^\mu$ and the Hamiltonian is given by $H = \gamma\cdot D$. We wish to diagonalize this Hamiltonian by using the same theorem, that is, find the transformation $i\gamma_5 \to \gamma\cdot D$. One recalls that in the treatment of Foldy and Wouthuysen $\gamma\cdot D$ was diagonalized by an infinite sequence of transformations. Each of these transformations was not only very complicated, but could only be determined after an evaluation of off-diagonal terms in the preceding order. Thus each approximation became rapidly more complicated to the extent that only a few of the lower-order off-diagonal terms could be eliminated. We will now show that a single transformation is sufficient to give exact diagonalization.

The operators $i\gamma_5$ and $\gamma\cdot D$ anticommute and both are Hermitian; however,

$$\begin{aligned}(\gamma\cdot D)^2 &= D_\alpha D^\alpha - eS^{\alpha\beta}F \\ &= -\partial_\alpha\partial^\alpha + e^2 A_\alpha A^\alpha - 2ieA^\alpha\partial_\alpha - eS^{\alpha\beta}F_{\alpha\beta},\end{aligned} \tag{30}$$

which is not unity. In order to invoke the theorem we must use an operator whose square is unity. We accomplished this in the free case by using the operator $\gamma\cdot P/m = \gamma\cdot P/(P_\mu P^\mu)^{1/2}$ instead of $\gamma\cdot P$. Here we use the same approach and use the theorem with $A = i\gamma_5$, and $B = \gamma\cdot D/M$ where M is the mass operator in the presence of an interaction, i.e., $M = [(\gamma\cdot D)^2]^{1/2}$. This operator is well defined (and diagonal) when acting on the eigenstates of $\gamma\cdot D$ and has eigenvalues which are the magnitude of the eigenvalues of $\gamma\cdot D$, and thus the eigenvalues of $\gamma\cdot D/M$ are ± 1 as required by the theorem. Furthermore, it follows that $\gamma\cdot D$ and M^{-1} commute. We assume that the state of the particle has no massless component and thus $\gamma\cdot D$ does not have a null eigenvalue. Thus M^{-1} for this case will be well defined.

Thus we subject the algebra to the unitary transformation

$$U = \exp(\theta i\gamma_5\gamma\cdot DM^{-1}). \tag{31}$$

For the Hamiltonian we get

$$H' = U\gamma\cdot DU^{-1} \tag{32a}$$

or

$$H' = \gamma\cdot D\cos 2\theta + i\gamma_5 M\sin 2\theta , \tag{32b}$$

which, with $\theta = \pi/4$, gives

$$U = \frac{1}{\sqrt{2}}(1 + i\gamma_5\gamma\cdot DM^{-1}) \tag{33}$$

and

$$H' = i\gamma_5 M , \tag{34}$$

where M is the mass operator.

Although we have formally diagonalized the Hamiltonian, a problem remains which was not present in the free-particle case where, with $H = i\gamma_5 m$, the operator m is defined on the mass eigenstates by $m = \int d^4k\, k\,|k\rangle\langle k|$, where the eigenvalue spectrum k^μ and the eigenstates $|k\rangle$ or equivalently $\langle y|k\rangle$ are exactly known. Thus the transformation can be exactly executed on the algebra as the commutators of m with the algebra are known. However, for the interacting case with $H|h\rangle = h|h\rangle$ the exact solutions, $\langle y|h\rangle$, are only known explicitly for those few cases where the Dirac equation is exactly solvable. Consequently the operator M, although still well defined, is not defined in the practical sense unless the problem is exactly solvable, i.e., unless the functions $\langle y|h\rangle$ are known.

One may use the procedure of Feynman[7] to define the square root of an operator Q which is the sum of two noncommuting operators:

$$Q = A + \lambda B , \tag{35}$$

$$Q^{-1/2} = \pi^{-1/2}\int_0^\infty e^{-Qu}u^{1/2}\,du , \tag{36a}$$

$$Q^{1/2} = (\pi^{-1/2}/2)\int_0^\infty (1 - e^{-Qu})u^{-3/2}\,du , \tag{36b}$$

with

$$e^{-(A+B)u} = e^{-Au} - \lambda u\int_0^1 e^{-AuS}Be^{-Au(1-S)}\,dS + \cdots . \tag{37}$$

One can show that this expansion is equivalent to a binomial expansion plus correction terms which arise from the noncommutativity of A and B. With $A = -\partial_\alpha\partial^\alpha$ and B as the rest of $(\gamma\cdot D)^2$, one sees that the noncommutative terms are of the form

$(P_\mu P^\mu)^{1/2} - [(P_\mu + R_\mu)(P^\mu + R^\mu)]^{1/2}$, where R^μ is the momentum component of the A^μ field. Such terms are very small if the field does not rapidly vary over distances of the order of the Compton wavelength of the particle. Thus for such fields one can use the binomial expansion to a good approximation. In the transformed representation with

$$M = (m^2 + B), \tag{38a}$$

$$B = e^2 A_\alpha A^\alpha - 2ieA^\alpha \partial_\alpha - eS^{\alpha\beta} F_{\alpha\beta}, \tag{38b}$$

one gets

$$H = i\gamma_5 M$$

$$\approx i\gamma_5\left(m + \frac{1}{2m} Q + \frac{1}{8m^3} Q^2 + \cdots\right). \tag{39}$$

Using either the exact Feynman expansion or the approximate binomial expansion for M and M^{-1} enables one to execute the transformation on the observables obtaining results in a series form. In conclusion, we have obtained the exact transformation in one step, but unless the functions $\langle y|h\rangle$ are exactly known one is forced to resort to a series expansion to find the transformed operators. Consequently, the Heisenberg equations and the separation into classical and nonclassical terms do not follow from this generalized Foldy-Wouthuysen transformation.

VI. HEISENBERG EQUATIONS

The proper-time Heisenberg equations for an operator Q are given by

$$\dot{Q}(\tau) = i[Q, H]. \tag{40}$$

In the general case of interactions, one obtains

$$\dot{X}^\mu = \gamma^\mu, \tag{41a}$$

$$\dot{D}^\mu = eF^{\mu\nu} \dot{X}_\nu, \tag{41b}$$

$$\dot{\gamma}^\mu = 2i\gamma^\mu H - 2D^\mu. \tag{41c}$$

As already pointed out, these equations do not show an obvious separation into classical and nonclassical portions. In the $H = i\gamma_5 M$ basis the transformed variables do not even have closed forms. It is easy to check that the mean variables do not obey equations of a closed form. We would like to have a closed form for the dynamical equations which exhibits the classical behavior in an obvious way.

Here the difficulty comes from the series form of M which must be used in obtaining the transformed algebra and also in the series form of H in the $H = i\gamma_5 M$ representation. However, the classical equations may easily be cast into many different forms, so perhaps the difficulty lies in trying to obtain the quantum-mechanical equation in the form

$$\dot{X}^\mu_{\rm cl} = m_0^{-1} D^\mu_{\rm cl}, \tag{42}$$

$$\dot{D}^\mu_{\rm cl} = eF^{\mu\nu} \dot{X}_{{\rm cl}\,\nu}. \tag{43}$$

As the mass operator which replaces m_0 does not generally commute with the other observables, one is tempted to look for other orderings of the mass operator in the dynamical equations. One easily checks that for an observable Q we get

$$[H, \dot{Q}]_+ = i[Q, H^2], \tag{44}$$

which is suggested by the usual antisymmetrization used in going to quantum mechanics. The usefulness of (44) is because H is the mass operator (apart from a sign) and the commutator $[Q, H^2]$ is both closed and classical in appearance as can be seen from the following equations which hold in any representation:

$$[H, \dot{X}^\mu]_+ = 2D^\mu, \tag{45a}$$

$$[H, \dot{D}^\mu]_+ = e(F^{\mu\alpha} D_\alpha + D_\alpha F^{\mu\alpha}) + eS_{\alpha\beta} F^{\alpha\beta,\mu}, \tag{45b}$$

$$[H, \dot{L}^{\mu\nu}]_+ = X^\mu F^\nu - F^\mu X^\nu, \tag{45c}$$

$$[H, \dot{\gamma}^\mu]_+ = 2eF^{\mu\alpha}\gamma_\alpha, \tag{45d}$$

$$[H, \dot{S}^{\mu\nu}]_+ = 2e(F^\mu{}_\alpha S^{\alpha\nu} - F^\nu{}_\alpha S^{\alpha\mu}), \tag{45e}$$

where

$$F^\mu \equiv [H, \dot{D}^\mu]_+. \tag{45f}$$

The first three of these may be compared to the classical equations rewritten in the form

$$[m_0, \dot{X}^\mu_{\rm cl}]_+ = 2D^\mu_{\rm cl}, \tag{46a}$$

$$[m_0, \dot{D}^\mu_{\rm cl}]_+ = e(F^{\mu\alpha} D_{{\rm cl}\alpha} + D_{{\rm cl}\alpha} F^{\mu\alpha}), \tag{46b}$$

$$[m_0, L^{\mu\nu}_{\rm cl}]_+ = X^\mu_{\rm cl} F^\nu - F^\nu X^\mu_{\rm cl}. \tag{46c}$$

Consequently we suggest that the form $[H, Q]_+$ is important and should be used to obtain the classical analogs. Equations (45a)–(45f) become more transparent in the Foldy representation where $H = i\gamma_5 M$ and for a Dirac particle $H = M$ when evaluated between Dirac states. The nonclassical contributions to $\dot{Q}$ for a general operator can be obtained by solving the $[H, \dot{Q}]_+$ for $\dot{Q}$.

VII. DISCUSSION

We have shown that the Dirac equation in the transformed representation is $i\gamma_5 M|\psi\rangle = m_0|\psi\rangle$ when $|\psi\rangle$ is a mass eigenstate. This equation is equivalent to the two equations

$$i\gamma_5|\psi\rangle = |\psi\rangle \tag{47}$$

and

$$M|\psi\rangle = m_0|\psi\rangle, \tag{48a}$$

or

$$M^2|\psi\rangle = m_0^2|\psi\rangle \ , \tag{48b}$$

which will be recognized as the equations proposed by Feynman and Gell-Mann.[8] What was not pointed out was the unitary equivalence to the Dirac theory and the fact that all operators and states must be subjected to the transformation. In particular it follows that the operator $1-i\gamma_5$ used to project out the electron neutrino states becomes $1+(\gamma\cdot p/M)$ in the new representation, and thus their argument for the form of the weak interaction is not clear. In fact these equations are space inversion-invariant as the inversion operator is also transformed into the new representation and as all commutation rules are invariant.

The difficulty in transforming the Dirac Hamiltonian in the usual form $i(\partial/\partial\tau)\psi = P^0\psi$ is due to the asymmetrical form which singles out the zeroth component.[2,3,4,6] Thus for time-dependent fields in the interacting case one must transform $i(\partial/\partial\tau)$ as shown in Eqs. (31) and (32) of Ref. 1. For the case of static fields with $A^0=0$, however, one can show that the remaining series of diagonal terms is in fact binomial expansion of $P^0=[(P-eA)^2+m^2]^{1/2}$. As the proper-time approach places all X^μ dependence on the same footing, one is able to achieve exact diagonalization even for time-dependent fields.

We have shown that the Dirac and the Feynman–Gell-Mann equations are unitarily equivalent by a covariant generalization of the Foldy-Wouthuysen transformation. Recently, Biedenharn, Han, and van Dam[9] have proposed an alternative to the Dirac equation which has been shown to be unitarily equivalent to the Dirac equation by de Vries and van Zanten.[10] The main feature of their equation is that it is linear in the momenta but with a space-time-dependent metric. This follows from

$$\begin{aligned}\psi^*\gamma^0\psi &\to \psi^* U^\dagger U\gamma^0 U^\dagger U\psi\\ &\to \psi'^*\gamma'^0\psi' \ ,\end{aligned} \tag{49}$$

where the new metric γ'^0 is space-time-dependent. This is in contrast to the space-time independence of our metric, as the transformation U which we use is unitary with respect to γ^0 (i.e., actually pseudounitary); thus

$$\begin{aligned}\psi^*\gamma^0\psi &\to \psi\gamma^0 U^{-1}U\psi\\ &\to \psi^* U^\dagger\gamma^0 U\psi\\ &\to \psi'^*\gamma^0\psi' \ .\end{aligned} \tag{50}$$

Their work differs from ours in that a different set of simultaneous observables is diagonalized. In fact, it is obvious that the Dirac equation may be cast into an infinite number of different forms by means of the infinite set of unitary transformations available. The new forms are useful only if other basic observables are diagonalized by the transformation or if the dynamical equations assume a more useful or physically interpretable form. In particular the Hamiltonian and various complete commuting sets of operators are obviously useful.

ACKNOWLEDGMENT

We have benefited from useful discussions with Professor Y. Aharonov and Professor James M. Knight.

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