Qualify Exam on Applied Math for Winter 2016

1. (a) Construct Lagrange's interpolation polynomial for the data given below.

   (b) Construct Newton's interpolation polynomial for the data shown. Without simplifying it, write the polynomial obtained in nested form for easy evaluation.

x	1	-1	3	2	-2
y	0	0	-1	-7	5

2. Construct the polynomial of best approximation of degree 2 in the L^2 norm to the function $f(x) = \sin x$ over $[-\pi/2, \pi/2]$.

3. Let $f : R \to R$ be a C^2 function with a root x_* such that neither f' nor f'' has a root. Prove that Newton's method converges to x_* for any initial guess $x_0 \in R$, and show the convergence rate.

4. Given that $f \in L^2(a, b)$, prove that there exists a unique polynomial $p_n \in P_n$ such that $||f - p_n||_2 = \min_{q \in P_n} ||f - q||_2$.

5. To solve ODE $y' = f(t, y)$ with initial value $y(0) = y_0$ we have several numerical methods. Show the formulas of the following methods and their corresponding properties.

   (a) Euler method: derive the local truncation error T_n using Taylor expansion. For ODE $y' = \lambda y$ with step size h, what is the domain of absolute stability?

   (b) Implicit Euler method: derive the local truncation error T_n using Taylor expansion. For ODE $y' = \lambda y$ with step size h, what is the domain of absolute stability?

   (c) Trapezoidal method: derive the local truncation error T_n using Taylor expansion.

   (d) Improved Euler (Runge-Kutta 2) method: derive the local truncation error T_n using Taylor expansion.

6. Using Taylor series expansions, derive the following two formulas for approximating the third derivative. Find their error terms. Which formula is more accurate?

$$f'''(x) \approx \frac{1}{h^3}[f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)]$$

$$f'''(x) \approx \frac{1}{2h^3}[f(x+2h) - 2f(x+h) + 2f(x-h) - f(x-2h)]$$

7. Consider the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$.

(a) Determine SVD of A.
(b) Determine QR factorization of A.
(c) What is the orthogonal projector of P onto range(A), and what is the image under P of the vector $(1, 0, -1)^*$?

8. Consider the $(n \times n)$ householder reflector matrix $F = I - 2\frac{uu^*}{u^*u}$, where $u \in C^n$. Show that F has eigenvalues $\lambda = -1$ and $\lambda = 1$.

9. Using the Gram-Schmidt iteration, find the QR factorization for:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

10. (a) For a matrix A, prove that A^*A is invertible if and only if A has linearly independent columns.
(b) Prove that if the matrix A has orthogonal columns, then $Ax = b$ has a unique least squares solution, and find this unique solution.