Analysis Qualifying Exam
August 2016

**Instructions:** Write your name legibly on each sheet of paper. Write only on one side of of each sheet of paper. Try to answer all questions. Questions 1–8 are each worth 10 points and question 9 is worth 20 points.
**Terminology:** Measurability and integrability on $\mathbb{R}$ or a measurable subset of it will always refer to the Lebesgue measure, except if otherwise specified. Lebesgue measure will be denoted by $m$, $dx$ or $dy$ depending on the context. If $A$ is a subset of $\mathbb{R}$ then $L_p(A)$ is considered with respect to the Lebesgue measure. You can quote without proof any of the standard theorems covered in Math 703-704, but do indicate why the relevant hypotheses hold.

1. Let $X$ be a metric space. Assume $K \subset X$ is a compact subset and $U \subset X$ is an open subset such that $K \subset U$. Prove there exists a $r > 0$ such that for all $x \in K$ we have $B(x, r) \subset U$. (Here $B(x, r)$ denotes the open ball with center $x$ and radius $r$.)

2. Let $E \subset \mathbb{R}$ be a bounded measurable set of positive measure. Prove that there exists $m, n \geq 1$ with $n \neq m$ such that

$$\left(E + \frac{1}{n}\right) \cap \left(E + \frac{1}{m}\right) \neq \emptyset.$$

   Conclude from this that there exists $x, y \in E$ with $x \neq y$ such that $x - y \in \mathbb{Q}$.

3. Let $0 \leq f_n$ be Lebesgue integrable functions on $\mathbb{R}$. Assume $f_n(x) \to 0$ a.e. on $\mathbb{R}$ as $n \to \infty$ and

$$\int \max(f_1(x), \cdots, f_n(x))\, dx \leq M$$

   for all $n \geq 1$. Prove that $\int f_n(x)\, dx \to 0$ as $n \to \infty$.

4. Let $E \subset [0, 1]$ be Lebesgue measurable and define for $x \in \mathbb{R}$

$$f_n(x) = n \int_0^{\frac{1}{n}} \chi_E(x + t)\, dt.$$

   Prove that

   (a) $0 \leq f_n(x) \leq 1$ for all $x \in \mathbb{R}$

(b) $f_n$ is a Lipschitz function for each $n$

(c) $f_n(x) \to \chi_E(x)$ a.e.

(d) $\|f_n - \chi_E\|_1 \to 0$.

5. Let $f \in L_1([0,1])$ and define $g(x) = \int_x^1 f(t)/t\,dt$ for $0 < x \le 1$. Prove that $g$ is absolutely integrable over $[0,1]$ and

$$\int_0^1 g(x)\,dx = \int_0^1 f(x)\,dx.$$

6. Let $f_n, f \in L^p(\mathbb{R})$ for some $2 \le p < \infty$. Asssume that $\|f_n - f\|_p \to 0$ as $n \to \infty$.

   (a) Prove that there exists a constant $M$ such that $\|f_n\|_p \le M$ for all $n \ge 1$.

   (b) Prove that $\|f_n^2 - f^2\|_{\frac{p}{2}} \to 0$ as $n \to \infty$.

7. Let $f$ be an entire function such that $|f(z)| \ge |z|$ for all $z \ne 0$. Prove that $f(z) = cz$ for some constant $c$ with $|c| \ge 1$.

8. Let $f$ be an entire function with power series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $z \in \mathbb{C}$.

   **a.** Assume that $f(\mathbb{R}) \subset \mathbb{R}$. Prove that $a_n \in \mathbb{R}$ for all $n \ge 0$.

   **b.** Assume now that $f(\mathbb{R}) \subset \mathbb{R}$ and $f(i\mathbb{R}) \subset i\mathbb{R}$. Prove that $f(-z) = -f(z)$ for all $z$.

9. True or False. Prove, or give a counterexample.

   **a.** There exist a measurable subset $E$ of $[0,1]$ such that for all $0 \le a < b \le 1$ we have $m(E \cap (a,b)) = \frac{b-a}{2}$.

   **b.** Let $f : [0,1] \to \mathbb{R}$ be a continuous function of bounded variation. Assume $f'(x) \ge 1$ a.e. and $f(0) \ge 0$. Then $f(x) \ge x$ for all $x \in [0,1]$.

   **c.** Let $f_n : [0,1] \to [0,\infty]$ be Lebesgue integrable functions such that $\int_0^1 f_n\,dx \to 0$. Then $f_n(x) \to 0$ a.e.

   **d.** If $z = 0$ is an essential singularity of $f$, then $z = 0$ is an essential singularity of $f^3$.

   **e.** There exists a function $f$ analytic on the $\{z \in \mathbb{C} : z \ne 0\}$ such that $f'$ has a simple pole at 0.