Team Round – University of South Carolina Math Contest, 2018

1. This is a team round. You have one hour to solve these problems as a team, and you should submit one set of answers for your team as a whole. Working together is, of course, encouraged.

2. Submit one answer per question, on the attached sheet. If you change your answer, make sure your earlier answer is clearly crossed out.

3. Drawings are not necessarily drawn to scale.

4. All answers should be in the form of a number – simplified as much as possible.

   Use your best judgment – for example, 1.75, $\frac{7}{4}$, $\left(\frac{31}{18}\right)^7$, $\frac{\sqrt{2}}{2}$, and $\frac{1}{\sqrt{2}}$ will all be considered to be in simplified form. $\sin(\pi/4)$, $\frac{\sqrt{8}}{4}$, and $5 + \frac{9}{5}$ will not be.

5. The questions are difficult. Do not be discouraged if you don't get them right away.

6. The competition consists of two parts. The first part has 10 independent questions, and the second part has 5 questions on a single theme. The questions in the second part are related, and working on earlier questions will help you with later questions in this part.

   There is also a 'take home question'. It's not part of the competition, and it doesn't 'count' for anything. But you might be interested in trying to solve it after you get home. You may submit solutions to `thorne@math.sc.edu` for an evaluation.

7. Your room will have a whiteboard or chalkboard, which you are encouraged to use. Markers or chalk, erasers, and blank paper will be provided by the proctors. (You should bring your own pencil or pen.) Please let your proctor know if your markers don't work or if you need additional markers, chalk, or paper.

8. No calculators, books, notes, or other tools are allowed.

9. Please let an exam proctor know if you have any questions about the exam, or if you require additional materials.

GOOD LUCK!

Answer Sheet for Team Round

Your School/Team Name: ____________________

1. ____________________
2. ____________________
3. ____________________
4. ____________________
5. ____________________
6. ____________________
7. ____________________
8. ____________________
9. ____________________
10. ____________________
11. ____________________
12. ____________________
13. ____________________
14. ____________________
15. ____________________

Part I.

1. Suppose that x, y, z are real numbers satisfying $x + y = 4$ and $|z + 1| = xy + 2y - 9$. What is $x + 2y + 3z$?

   Solution. Note that

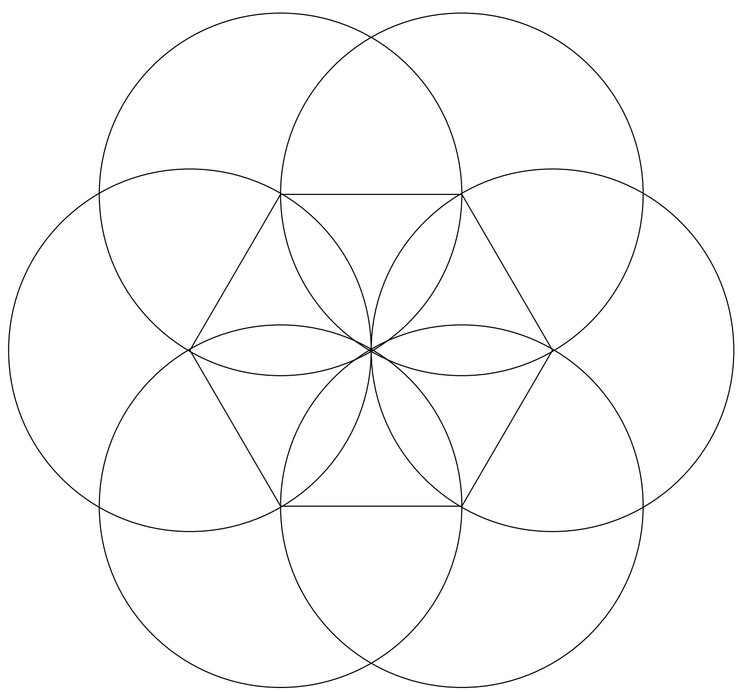
   $$|z+1| = xy + 2y - 9 = (4-y)y + 2y - 9 = 6y - y^2 - 9 = -(y-3)^2.$$

   The left side is nonnegative and the right side is nonpositive. So both sides must be equal, with $z = -1$, $y = 3$, and $x = 4 - 3 = 1$. We have $x + 2y + 3z = 1 + 2 \cdot 3 + 3(-1) = 4$.

2. Let H be a regular hexagon with side length 1. Circles of radius 1 are drawn centered at each of the vertices of H.

   Let A be the subset of points of H which are in or on at least three of the circles. What is the area of A?

   Solution. The picture looks like this:



D

The area of the hexagon is $6 \cdot \frac{\sqrt{3}}{4} = \frac{3\sqrt{3}}{2}$. Each point is in either two or three circles, let x be the area within three circles, so that $\frac{3\sqrt{3}}{2} - x$ is the area within two.

Each of the six circles has area $\frac{\pi}{3}$ within the hexagon. So we have

$$2\pi = 2\left(\frac{3\sqrt{3}}{2} - x\right) + 3x = x + 3\sqrt{3}.$$

So $x = 2\pi - 3\sqrt{3}$.

3. If n is a positive integer, $\sigma(n)$ denotes the sum of the positive integer divisors of n. For example, $\sigma(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$.

   It is true that $\sigma(\sigma(88)) = \sigma(\sigma(90)) = 546$, and there is exactly one positive integer $n \leq 100$ for which $\sigma(\sigma(n)) > 546$. What is $\sigma(\sigma(n))$ for this value of n?

   Solution. This question rewards educated guesswork, numerical experimentation, and careful computation. If you compute various values of $\sigma(n)$, you will notice that $\sigma(n)$ is larger if (1) n is larger, and n has lots of small prime factors. For example, if n is divisible by 2, 3, and 4 (equivalently, divisible by 12), then we will have

$$\sigma(n) = n + \frac{n}{2} + \frac{n}{3} + \frac{n}{4} + \cdots > \frac{13}{12}n.$$

   So a logical first guess is $n = 96$, the largest $n \leq 100$ divisible by 12. Indeed, we have

$$\sigma(96) = 96 + 48 + 32 + 24 + 16 + 12 + 8 + 6 + 4 + 3 + 2 + 1 = 252,$$

$$\sigma(252) = 252+126+84+63+42+36+28+21+18+14+12+9+7+6+4+3+2+1 = 728.$$

4. Simplify:

$$\frac{1}{\log_2(2018!)} + \frac{1}{\log_3(2018!)} + \frac{1}{\log_4(2018!)} + \cdots + \frac{1}{\log_{2018}(2018!)}.$$

   Solution. This is

$$\frac{\log(2)}{\log(2018!)} + \frac{\log(3)}{\log(2018!)} + \cdots + \frac{\log(2018)}{\log(2018!)} = \frac{\log(2018!)}{\log(2018!)} = 1.$$

5. Suppose that x and y are two real numbers satisfying $x + y = 3$ and $\frac{1}{x+y^2} + \frac{1}{x^2+y} = \frac{1}{2}$. What is $x^5 + y^5$?

   Solution. Clearing denominators, we have

$$2(x^2 + y + x + y^2) = (x + y^2)(x^2 + y),$$

   or

$$6 + 2(x^2 + y^2) = x^3 + y^3 + xy + (xy)^2.$$

   Let $u = xy$; we have

$$9 = (x + y)^2 = x^2 + y^2 + 2u,$$

$$27 = (x + y)^3 = x^3 + y^3 + 3xy(x + y) = x^3 + y^3 + 9u.$$

   We substitute these expressions into the above, getting

$$6 + 2(9 - 2u) = 27 - 9u + u + u^2,$$

   or

$$u^2 - 4u + 3 = 0,$$

so $u = 1$ or $u = 3$. The solution $u = 3$ is extraneous: we would have $x^3 + y^3 = 0$, hence $x = -y$, but this contradicts that $x + y = 3$. So $u = xy = 1$, and we thus see that $x^2 + y^2 = 7$ and $x^3 + y^3 = 18$.

Although we can solve for x, it's simpler not to. We have

$$243 = (x+y)^5 = x^5 + y^5 + 5xy(x^3 + y^3) + 10(xy)^2(x + y) = x^5 + y^5 + 5 \cdot 1 \cdot 18 + 10 \cdot 1 \cdot 3,$$

so $x^5 + y^5 = 123$.

6. In the (infinite) decimal expansion of $1/999998$, what is the 98th digit after the decimal point?

Solution. (Corrected!) We have

$$\frac{1}{999998} = .000001000002000004000008000016000032000064\ldots$$

The pattern is evident: the powers of 2 appear every six digits. This is because

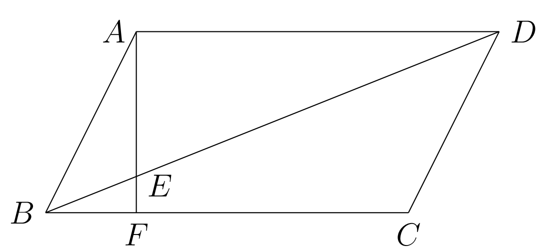
$$\frac{1}{999998} = .000001 \times \frac{1}{1 - .000002} = .000001 \times (1 + .000002 + .000002^2 + .000002^3 + \cdots)$$

The pattern continues until the powers of 2 become larger than one million. Since $98 = 6 \cdot 16 + 2$, the 98th digit is the second digit in the 17th grouping. Note that the 17th grouping consists of the six digits of 2^{16} (since the first consists of the six digits $000001 = 2^0$). Therefore, the solution is the ten thousands digit of

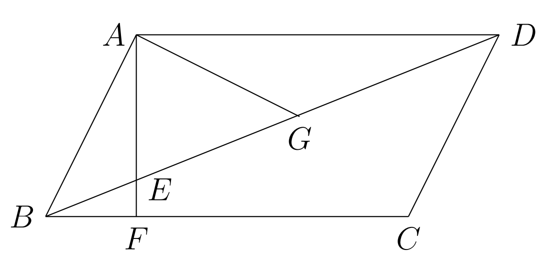
$$2^{16} = 066536,$$

or 6.

7. In the parallelogram $ABCD$ depicted below, we have $\angle ABC = 72°$, $AF \perp BC$, AF intersects BD at E, and $\overline{ED} = 2\overline{AB}$. Compute the measure of $\angle AED$.



Solution. Connect A to the midpoint G of ED:



We have $\overline{BA} = \overline{AG} = \overline{GD} = \overline{EG}$. Writing $x = m\angle AED$, we have $m\angle ADG = m\angle DBC = 90 - x$, so that $m\angle ABD = 72 - (90 - x) = x - 18$. We also have $m\angle AGE = 180 - 2x = m\angle ABG$. So $x - 18 = 180 - 2x$, hence $x = 66$.

8. You and one opponent play a game as follows. There is a pile of pennies, and on your turn you must take exactly one, two, or six of them. The player forced to take the last penny loses.

   Call a positive integer n *good* if, when the pile has n pennies in it and it is your turn, you can force a win. For example, 7 is good: you may take 6 pennies and thereby force your opponent to take the last one.

   How many positive integers $n \leq 2018$ are good?

   Solution. We begin enumerating the good small integers up to 7.

   - 1 is bad;
   - 2 is good (take one penny);
   - 3 is good (take two pennies);
   - 4 is bad (you must leave your opponent with 2 or 3);
   - 5 is good (take one penny);
   - 6 is good (take two pennies);
   - 7 is good (take six pennies);

   Continuing up to 14:

   - 8 is bad (you must leave your opponent with 2, 6, or 7)
   - 9 is good (take one penny);
   - 10 is good (take two pennies);
   - 11 is bad (you must leave your opponent with 5, 9, or 10);
   - 12 is good (take one penny);
   - 13 is good (take two pennies);
   - 14 is good (take six pennies);

   The same analysis in the second list of options shows more: for any $n \geq 8$, n is good if and only if $n - 7$ is. (Here the results for 8 through 14 depended only on whether 1 through 7 are good; 15 through 21 will depend on 8 through 14 in the same way; and so on.)

   So the sequence of good numbers is periodic modulo 7, only depending on the remainder when n is divisible by 7. We have $2018 = 7 \cdot 288 + 2$. There are $5 \cdot 288 = 1440$ good integers through $7 \cdot 288$, and 2018 is good also, so the answer is 1441.

9. How many positive integers $n \leq 2018$ can be written in the form $a^3 + b^3$, where a and b are positive integers?

   Solution. The answer is 68. We list the pairs (a, b) with $a \geq b$ that produce $a^3 + b^3 \leq 2018$. These are:

- $a = 12$, $b \le 6$;
- $a = 11$, $b \le 8$;
- $a \le 10$, $b \le 10$.

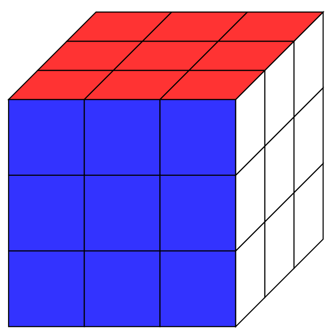
The total number of pairs is

$$6 + 8 + (10 + 9 + 8 + \cdots + 1) = 6 + 8 + 55 = 69.$$

As Ramanujan famously observed when entering a taxicab with the number 1729, it is the smallest number that can be written as the sum of two cubes in different ways, $1729 = 10^3 + 9^3 = 12^3 + 1^3$. The remaining (a, b) all generate unique values of $a^3 + b^3$, so the answer is $69 - 1 = 68$.

Note that there is a shortcut to checking that the remaining (a, b) all generate unique values of $a^3 + b^3$. The numbers 1^3 through 10^3 all end in different digits, so most possibilities can be ruled out by looking at the last digit alone.

10. Consider a Rubik's cube, where each of the six faces has sixteen *corner points*. For example, the corner points on three of the faces occur at the intersections of the line segments in the diagram below:



Points interior to the cube are not considered corner points.

How many distinct (nonzero) distances are there between pairs of corner points?

Solution. It is easier and equivalent to compute the number of distinct *squared* distances, so we will do that.

We anchor the cube in 3-space, so the bottom left corner is at $(x, y, z) = (0, 0, 0)$ and the top right corner is $(3, 3, 3)$. The corner points are all of the form (x, y, z) with $0 \le x, y, z \le 3$, where at least one of x, y, and z is either 0 or 3. The squared distance between (x, y, z) and (x', y', z') is $(x - x')^2 + (y - y')^2 + (z - z')^2$, which must be of the form $a^2 + b^2 + c^2$, with $0 \le (a, b, c) \le 3$.

The first step (and this is not obvious) is to note that any such combination of a, b, and c, other than $(0, 0, 0)$, can occur. For example, given such a combination of (a, b, c), start with both points on the bottom right corner in the diagram; move one point a units to the left, and the other point b units up and c units to the back.

We are therefore reduced to counting the number of integers between 1 and 27 which can be written in the form $a^2 + b^2 + c^2$, with $0 \leq (a, b, c) \leq 3$. These are not too difficult to enumerate; we have

$$1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 17, 18, 19, 22, 27.$$

There are 18 possibilities total.

Part II. The following questions concern the game *Let 'em Roll*, which appears on the TV show *The Price Is Right*.

You are given five six-sided dice, and you can win up to \$7,500 in cash, or a \$15,000 car. Each die has a picture of a car on three faces, and has dollar amounts of \$500, \$1,000, and \$1,500 on the other three faces. At the end of the game, if all five dice have a car showing face up, you win the car. If any dice show a monetary amount, you win the total amount of money showing on the dice (with cars counting as nothing).

You get three rolls[1] of the dice. After the first and second rolls, you may either accept the results and end the game, or continue. If you continue, the dice showing a car are taken out of play, and you reroll only those dice showing money.

11. What is the probability you win the car on the first roll of the dice?

    Solution. Each die has to show a car, of which there is a 1 in 2 chance, so $(1/2)^5 = 1/32$.

12. If you play for the car, what is the probability you win the car on the first or second roll of the dice?

    Solution. For each die, the probability that you fail to roll a car on either the first or second attempt is $1/4$, so the probability of getting a car for each die is $3/4$. So the final answer is $(3/4)^5 = 243/1024$. (Note that $(3/4)^5$ is an acceptable answer.)

13. If you play for the car, what is the probability you win it?

    Solution. Similar to the last one, $(7/8)^5$.

14. Suppose that after two rounds you have not rolled any cars, but all five dice show \$1,500. You give up the \$7,500 in prize money and reroll all five dice on the third round.

    What is the *expected value* of your roll – that is, the average amount you expect to win, counting the car as equivalent to \$15,000 in cash? *(Answer as an exact number of dollars and cents.)*

    Solution. There is a 1 in 32 chance that you win the car. Otherwise, you will win some money – the average amount of money on each die is 500, and there are five dice, so

$$\frac{15000}{32} + 500 \cdot 5 = 2968.75.$$

15. Let $S \subseteq \{0, 1, 2, 3, 4, 5\}$ be the set of integers n for which the following statement is true:

    Suppose that after two rounds you have rolled $5 - n$ dice showing cars, and n dice each showing \$1,500. If you choose to reroll the n dice showing \$1,500, then the expected value of your roll is larger than $1500n$.

    What is the sum of the elements of S?

[1]On the actual game show, you get three rolls only if you price some small grocery items correctly. Assume that you do.

Solution. If you reroll n dice, the question is whether

$$\frac{15000}{2^n} + 500n > 1500n,$$

or (simplifying) $\frac{15000}{2^n} > 1000n$, or $n \cdot 2^n < 15$. This is true for $n \in \{0, 1, 2\}$, so the answer is 3.

Take Home Question. Don't work on this now, it's not part of the competition. If you solve it after you get home, you may submit your solution to `thorne@math.sc.edu` for an evaluation.

Describe and justify, as well as you can, an optimal or near-optimal strategy for playing this game. Assume that you value the car equivalently to $15,000 cash, and play to maximize your expected value.

In particular, under what situations does it make sense not to reroll?

Answer Key.

1. 4
2. $2\pi - 3\sqrt{3}$
3. 728
4. 1
5. 123
6. 6
7. 66
8. 1441
9. 68
10. 18
11. $\frac{1}{2^5}$ or $\frac{1}{32}$
12. $\left(\frac{3}{4}\right)^5$ or $\frac{243}{1024}$
13. $\left(\frac{7}{8}\right)^5$ or $\frac{16807}{32768}$
14. 2968.75
15. 3