

State High School Mathematics Tournament

University of South Carolina

February 3, 2018



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Tournament Round 1 – Rules



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- ▶ **Two winners will be selected this round: the participants earning three points in the fewest number of questions.**
- ▶ There will be a tiebreaker if needed.



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Question 1-1

If

$$\log_2(x) + \log_8(4x) = 10,$$

what is x ?



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Solution 1-1

Answer: 128.



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Solution 1-1

Answer: 128.

Solution. We have

$$\log_2(x) + \frac{2}{3} + \frac{\log_2(x)}{3} = 10,$$

hence

$$\frac{4}{3} \log_2(x) = 10 - \frac{2}{3} = \frac{28}{3},$$

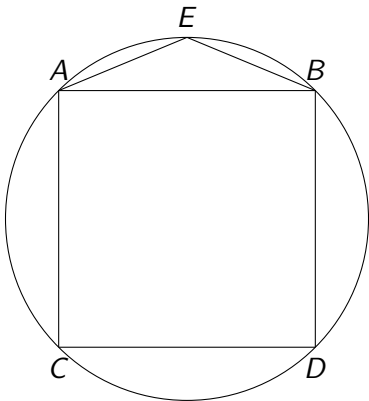
so $\log_2(x) = 7$ and $x = 2^7 = 128$.



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Question 1-2

A square is inscribed in a circle of radius 1 as follows:

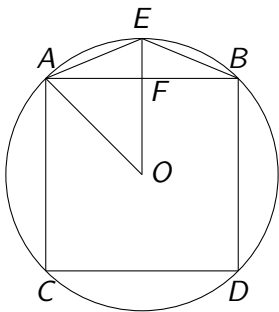


If $\overline{AE} = \overline{BE}$, find the area of $\triangle AEB$.



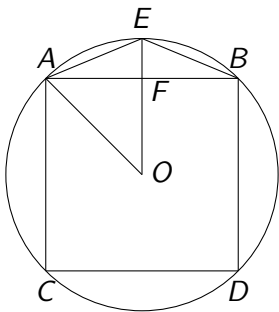
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Solution 1-2



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Solution 1-2

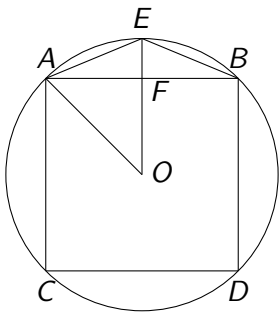


We have $\overline{AO} = \overline{EO} = 1$, $\overline{AB} = \sqrt{2}$,



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Solution 1-2

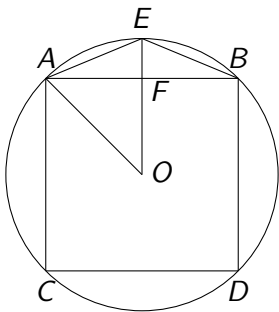


We have $\overline{AO} = \overline{EO} = 1$, $\overline{AB} = \sqrt{2}$, $\overline{EO} \perp \overline{AB}$,



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Solution 1-2

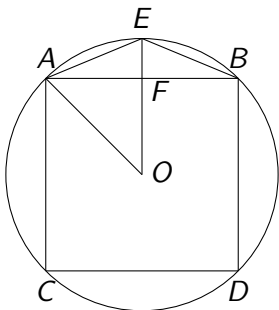


We have $\overline{AO} = \overline{EO} = 1$, $\overline{AB} = \sqrt{2}$, $\overline{EO} \perp \overline{AB}$, $\overline{OF} = \frac{\sqrt{2}}{2}$,



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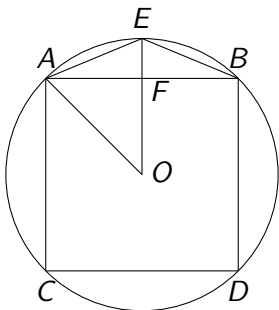


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 $\overline{EF} = 1 - \frac{\sqrt{2}}{2}$.



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Solution 1-2



We have $\overline{AO} = \overline{EO} = 1$, $\overline{AB} = \sqrt{2}$, $\overline{EO} \perp \overline{AB}$, $\overline{OF} = \frac{\sqrt{2}}{2}$, $\overline{EF} = 1 - \frac{\sqrt{2}}{2}$. So $\triangle AEB$ has area

$$\frac{1}{2} \cdot \sqrt{2} \cdot \left(1 - \frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2} - 1}{2}.$$



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Question 1-3

An empty pasture starts with a certain amount of grass. The grass grows at a constant rate, and cows eat grass at a constant rate.



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- ▶ If you allow 16 cows onto the pasture, they will eat all the grass in 10 days.



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- ▶ If you allow 16 cows onto the pasture, they will eat all the grass in 10 days.
- ▶ If you instead allow only 10 cows onto the pasture, they will eat all the grass in 22 days.



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- ▶ If you allow 16 cows onto the pasture, they will eat all the grass in 10 days.
- ▶ If you instead allow only 10 cows onto the pasture, they will eat all the grass in 22 days.

If you allow 25 cows onto the pasture, how long will it take them to eat all the grass?



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Solution 1-3

Answer. 5.5 days.



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Answer. 5.5 days.

- ▶ Let $f(t)$ measure the amount of grass in the pasture after t days, as a function of t . As units we choose the amount of grass one cow eats in one day.



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- ▶ Let $f(t)$ measure the amount of grass in the pasture after t days, as a function of t . As units we choose the amount of grass one cow eats in one day.
- ▶ By assumption, we can write $f(t) = a + bt$ for some a and b .



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- ▶ We have $a + 10b = 160$ and $a + 22b = 220$. Solving, we get $b = 5$ and $a = 110$.



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- ▶ By assumption, we can write $f(t) = a + bt$ for some a and b .
- ▶ We have $a + 10b = 160$ and $a + 22b = 220$. Solving, we get $b = 5$ and $a = 110$.
- ▶ Let d be the number of days it will take for 25 cows to eat the grass. Then we have $110 + d \cdot 5 = 25d$.



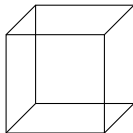
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- ▶ Let d be the number of days it will take for 25 cows to eat the grass. Then we have $110 + d \cdot 5 = 25d$.
- ▶ Solving for d , we obtain $d = 5.5$.



Question 1-4

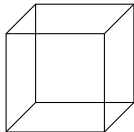
A 3-dimensional cube has 12 edges:



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Question 1-4

A 3-dimensional cube has 12 edges:



How many edges does a 7-dimensional cube have?



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Answer. 448.

Solution. There are $2^7 = 128$ vertices; each is connected by an edge to 7 other vertices, so

$$128 \cdot 7 \cdot \frac{1}{2} = 448.$$



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Question 1-5

If you expand $(x + 2y)^6$, what is the sum of all the coefficients?



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Solution 1-5

Answer. 729.

Solution 1. We have

$$(x+2y)^6 = x^6 + 6 \cdot 2x^5y + 15 \cdot 4x^4y^2 + 20 \cdot 8x^3y^3 + 15 \cdot 16x^2y^4 + 6 \cdot 32xy^5 + 64y^6,$$



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$$1 + 6 \cdot 2 + 15 \cdot 4 + 20 \cdot 8 + 15 \cdot 16 + 6 \cdot 32 + 64 = 1 + 12 + 60 + 160 + 240 + 192 + 64 = 729.$$



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Solution 1-5

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Solution 2. Adding the coefficients is the same as substituting $x = y = 1$, and



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Solution 1-5

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Solution 1. We have

$$(x+2y)^6 = x^6 + 6 \cdot 2x^5y + 15 \cdot 4x^4y^2 + 20 \cdot 8x^3y^3 + 15 \cdot 16x^2y^4 + 6 \cdot 32xy^5 + 64y^6,$$

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Solution 2. Adding the coefficients is the same as substituting $x = y = 1$, and

$$(1 + 2)^6 = 3^6 = 729.$$



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Question 1-6

Consider the set of integers that can be written in the form $y^2 - x^2$, where x and y are positive integers with $1 \leq x \leq y \leq 10$.



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Question 1-6

Consider the set of integers that can be written in the form $y^2 - x^2$, where x and y are positive integers with $1 \leq x \leq y \leq 10$.

How many of them are prime?



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Answer. 7.

Solution. We have

$$y^2 - x^2 = (y - x)(y + x),$$

so if this is prime then $y = x + 1$ and $y^2 - x^2 = 2x + 1$.



Answer. 7.

Solution. We have

$$y^2 - x^2 = (y - x)(y + x),$$

so if this is prime then $y = x + 1$ and $y^2 - x^2 = 2x + 1$.

We therefore count the set of prime integers of the form $2x + 1$ (i.e., odd) between 3 and 19, which is

$$\{3, 5, 7, 11, 13, 17, 19\}.$$



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Question 7

What is the last digit of 3^{2018} ?



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Solution 7

Answer. 9.

Solution. Notice that $3^4 = 81$, with last digit 1.



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Solution 7

Answer. 9.

Solution. Notice that $3^4 = 81$, with last digit 1. Since

$$3^{2018} = 3^{4 \cdot 504 + 2} = (81)^{504} \cdot 9,$$

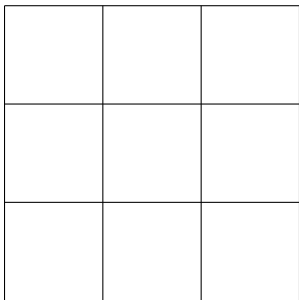
the last digit of 3^{2018} is $1^{504} \cdot 9 = 9$.



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Question 8

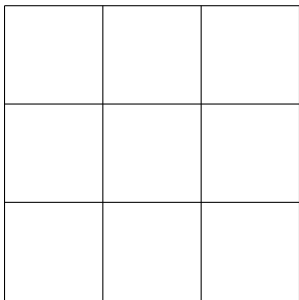
Consider (again) a Rubik's cube, where each of the six faces has sixteen *corner points*, illustrated by the intersections of the line segments as follows:



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Question 8

Consider (again) a Rubik's cube, where each of the six faces has sixteen *corner points*, illustrated by the intersections of the line segments as follows:



How many corner points are there on the cube total?



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Solution 8

Answer. 56.

Solution. On each face, there are 16 corner points. Of these:



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Solution 8

Answer. 56.

Solution. On each face, there are 16 corner points. Of these:

- ▶ 4 are on that face alone, and $4 \cdot 6 = 24$;



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Solution 8

Answer. 56.

Solution. On each face, there are 16 corner points. Of these:

- ▶ 4 are on that face alone, and $4 \cdot 6 = 24$;
- ▶ 8 are shared with one other face, and $8 \cdot 3 = 24$;



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Answer. 56.

Solution. On each face, there are 16 corner points. Of these:

- ▶ 4 are on that face alone, and $4 \cdot 6 = 24$;
- ▶ 8 are shared with one other face, and $8 \cdot 3 = 24$;
- ▶ 4 are shared with two other faces, and $4 \cdot 2 = 8$.



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Solution 8

Answer. 56.

Solution. On each face, there are 16 corner points. Of these:

- ▶ 4 are on that face alone, and $4 \cdot 6 = 24$;
- ▶ 8 are shared with one other face, and $8 \cdot 3 = 24$;
- ▶ 4 are shared with two other faces, and $4 \cdot 2 = 8$.

$$24 + 24 + 8 = 56.$$



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Question 9

The squares of three consecutive positive integers are added, to obtain 770.

What is the smallest of these integers?



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Solution 9

Answer. 15,

$$15^2 + 16^2 + 17^2 = 225 + 256 + 289 = 770.$$



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Solution 9

Answer. 15,

$$15^2 + 16^2 + 17^2 = 225 + 256 + 289 = 770.$$

Note that if n denotes the *middle* number, we have

$$(n-1)^2 + n^2 + (n+1)^2 = (n^2 - 2n + 1) + n^2 + (n^2 + 2n + 1) = 3n^2 + 2,$$

so $3n^2 = 768$, $n^2 = 256$, and $n = 16$.



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Question 10

You flip two coins. One is fair; the other is weighted and is more likely to come up heads than tails.

If the probability of flipping at least one heads is 80%, what is the probability of flipping both heads?



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Solution 10

Answer. $\frac{3}{10}$.

Solution. Let p be the probability that the weighted coin comes up heads.

The probability of flipping no heads is

$$\frac{1}{2}(1 - p) = \frac{1}{5},$$

so $1 - p = \frac{2}{5}$ and $p = \frac{3}{5}$. The probability of flipping two heads is thus

$$\frac{1}{2} \times \frac{3}{5} = \frac{3}{10}.$$



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Question 11

What is

$$1 - 2 + 3 - 4 + 5 - \cdots + 2017 - 2018?$$



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Solution 11

Answer. -1009 . Write it as

$$(1 - 2) + (3 - 4) + (5 - 6) + \cdots + (2017 - 2018),$$

which is -1 added 1009 times.



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Question 12

There are unique integers a and b for which



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Question 12

There are unique integers a and b for which

$$(1 + \sqrt{5})^3 = a + b\sqrt{5}.$$

What is $a + b$?



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Answer. 24.



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Answer. 24. We have

$$(1 + \sqrt{5})^3 = 1 + 3\sqrt{5} + 3(\sqrt{5})^2 + (\sqrt{5})^3 = 16 + 8\sqrt{5}.$$



Question 13

How many digits are in the base 10 number 20^{18} ?



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Solution 13

Answer: 24.

Solution. We have

$$20^{18} = 262144000000000000000000,$$

which is 2^{18} with 18 zeroes after it.



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Solution 13

Answer: 24.

Solution. We have

$$20^{18} = 262144000000000000000000,$$

which is 2^{18} with 18 zeroes after it.

$$2^{18} = 2^{10}2^8 = 1024 \cdot 256 \sim 1000 \cdot 250 = 250000,$$

with six digits, and $18 + 6 = 24$.



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