XXXII High School Math Contest

University of South Carolina

February 3rd, 2018

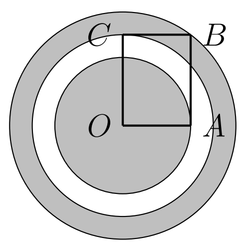
Problem 1. For what values of the real number a, does the quadratic equation $x^2+ax+a=0$ have two real zeros, with one of these zeros positive and the other zero negative?

(a) $a<0$ (b) $0<a<4$ (c) $a>1$ (d) $a>4$ (e) no such values exist

Answer: (a)

Solution: Since the quadratic equation has two real zeros, its discriminant a^2-4a is positive, so $a<0$ or $a>4$. Also, the graph of the parabola x^2+ax+a opens upward, intersects the x–axis at a negative number and at a positive number, thus its y–intercept a is negative. □

Problem 2. Given a rectangle $OABC$ as in the figure below, with $OA=a$ and $OC=c$, three concentric circles with radii $\overline{OA}$, $\overline{OB}$ and $\overline{OC}$ are drawn. Compute the difference between the area of the inner circle with radius $\overline{OA}$ and the area of the annulus generated by the concentric circles with radii $\overline{OB}$ and $\overline{OC}$. (An annulus generated by two concentric circles is the region between the circles.)



(a) 0 (b) $\pi a^2-\frac{\pi}{2}c^2$ (c) $\frac{\pi}{2}(a-c)^2$ (d) $\frac{\pi}{3}(a-c)^2$ (e) $\pi c^2-\frac{\pi}{2}a^2$

Answer: (a)

Solution: Denote $\overline{OB}=b$. The area of the inner circle is πa^2. The area of the annulus is $\pi(b^2-c^2)$. Since $OABC$ is a rectangle, $a^2+c^2=b^2$. □

Problem 3. How many real solutions does the following equation have?

$$x^2+\sqrt{x-2}=4+\sqrt{4-2x}$$

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

Answer: (b)

Solution: The square roots are defined only when $x - 2 \geq 0$ and $4 - 2x \geq 0$. The two inequalities are satisfied only when $x = 2$. Since $x = 2$ is a solution, the equation has exactly one real solution. $\square$

Problem 4. Let $x = \log_{10}(81)$ and $y = \log_{10}(25)$. If one expresses $\log_{10}(6)$ as $ax + by + c$ where a, b, and c are rational numbers, then what is $a + b + c$?

(a) $-\frac{7}{4}$ (b) $-\frac{5}{6}$ (c) $\frac{3}{4}$ (d) $\frac{5}{6}$ (e) $\frac{7}{4}$

Answer: (c)

Solution: Since $81 = 3^4$, we get $\log_{10}(3) = x/4$. Also, $25 = 5^2$, so we obtain $\log_{10}(5) = y/2$. Note that $\log_{10}(5) + \log_{10}(2) = 1$, so $\log_{10}(2) = 1 - y/2$. Thus, $\log_{10}(6) = \log_{10}(2) + \log_{10}(3) = 1 - y/2 + x/4$. $\square$

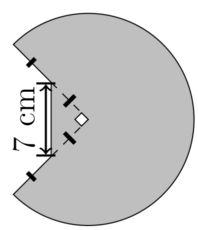
Problem 5. If x is a real number such that $x - x^{-1} = 3$, what is the value of $x^3 - x^{-3}$?

(a) 11 (b) $8 + 5\sqrt{13}$ (c) 27 (d) $5 + 8\sqrt{13}$ (e) 36

Answer: (e)

Solution: Cubing the equation $x - x^{-1} = 3$ we get $27 = x^3 - 3x + 3x^{-1} - x^{-3} = x^3 - x^{-3} - 9$. $\square$

Problem 6. The shaded region below is a union of three quarters of a circle and a right triangle whose vertices are the center of the circle and the two midpoints of the the shown radii. Find its area.



(a) $\frac{147\pi}{2} + \frac{49}{4}$ (b) $147\pi + \frac{49}{4}$ (c) $147\pi + \frac{49}{2}$ (d) $196\pi + \frac{49}{4}$ (e) $196\pi + \frac{49}{2}$

Answer: (a)

Solution: Both legs of the right triangle have a length of $7/\sqrt{2}$, thus the area of the triangle is $49/4$. The radius of the circle is $7\sqrt{2}$, so the area of the region is $(3/4)(98\pi) + 49/4$. $\square$

Problem 7. Consider all integers a such that both zeros of the following quadratic equation are integers.

$$x^2 - ax + 2a = 0$$

What is the sum of all such integers a?

(a) 7 (b) 8 (c) 10 (d) 12 (e) 16

Answer: (e)

Solution: Since both roots of the quadratic equation are integers and its coefficients are integers, then its discriminant $a^2 - 8a$ must be a square of an integer, say $a^2 - 8a = b^2$ for some integer b. So, $(a-4)^2 - 16 = b^2$, $(a-4)^2 - b^2 = 16$, and $(a-4-b)(a-4+b) = 16$. Hence, the possible values for $a - 4 - b$ are $-16, -8, -4, -2, -1, 1, 2, 4, 8, 16$ and the possible values for $a - 4 + b$ are $-1, -2, -4, -8, -16, 16, 8, 4, 2, 1$, respectively. Thus, the possible values for $2(a-4) = (a-4-b) + (a-4+b)$ are $-10, -8, 8, 10$ (note that $2(a-4)$ is even). Therefore, a has to be one of $-1, 0, 8, 9$. One checks directly that for each of these values of a, all zeros of the quadratic equation indeed are integers. □

Problem 8. Let $a, b, c > 1$ and $x > 1$ so that $\log_{ab}(x) = 9$, $\log_{bc}(x) = 18$ and $\log_{abc}(x) = 8$. Determine $\log_{ac}(x)$.

(a) 9 (b) 12 (c) 18 (d) 24 (e) 36

Answer: (b)

Solution: The given equations can be rewritten in the form $(ab)^9 = x$, $(bc)^{18} = x$, and $(abc)^8 = x$. Therefore, $ab = x^{1/9}$, $bc = x^{1/18}$, and $abc = x^{1/8}$. Thus, $c = (abc)/(ab) = x^{1/72}$, $a = (abc)/(bc) = x^{5/72}$, so $ac = x^{1/12}$. □

Problem 9. The $x-$ and $y-$coordinates of two different points $(\sqrt{\pi}, a)$ and $(\sqrt{\pi}, b)$ satisfy the following equation:

$$y^2 + x^4 = 2x^2y + 1.$$

What is $|a - b|$?

(a) 1 (b) $\frac{\pi}{2}$ (c) 2 (d) $\sqrt{1+\pi}$ (e) $1 + \sqrt{\pi}$

Answer: (c)

Solution: The equation can be written in the form $(y - x^2)^2 = 1$, or $y - x^2 = \pm 1$. Thus, $a - \pi = \pm 1$, and $b - \pi = \pm 1$. Since a and b are distinct, $a - b = \pm 2$. □

Problem 10. Let S be a sphere of radius 8 and let C be a cube whose eight vertices lie on the surface of S. What is the volume of C?

(a) $\frac{512\sqrt{3}}{3}$ (b) $\frac{512\sqrt{6}}{3}$ (c) $\frac{1024\sqrt{3}}{3}$ (d) $\frac{4096\sqrt{3}}{9}$ (e) $1024\sqrt{2}$

Answer: (d)

Solution: The centers of the sphere and the cube coincide, so the longest diagonal of the cube is the diameter of the sphere, hence has length 16. Now, if a is the side of the cube, its longest diagonal has length $a\sqrt{3}$. Thus, $a = 16/\sqrt{3}$. □

Problem 11. A person rolls four fair six-sided dice. What is the probability that the person rolls exactly one 1 and exactly one 2?

(a) $\frac{4}{27}$ (b) $\frac{16}{81}$ (c) $\frac{302}{6^4}$ (d) $\frac{500}{6^4}$ (e) $\frac{65}{81}$

Answer: (a)

Solution: The total number of outcomes is 6^4. Let us count the number of favorable outcomes. There four ways to pick the die which shows 1. There are three ways to pick one the remaining dice to show 2. Finally, each of the remaining two dice can show any of the numbers $3, 4, 5, 6$. The number of favorable outcomes is $4 \cdot 3 \cdot 4 \cdot 4$. □

Problem 12. Suppose $x \neq y$ and two sequences of numbers

$$x, a_1, a_2, a_3, y, \qquad \text{and} \qquad b_1, x, b_2, b_3, y, b_4$$

are both arithmetic sequences. What is the value of $\dfrac{b_4 - b_3}{a_2 - a_1}$?

(a) 2/3 (b) 4/3 (c) 5/3 (d) 7/3 (e) 8/3

Answer: (e)

Solution: Let d_1 be the difference of the first arithmetic sequence, and let d_2 be the difference of the second arithmetics sequence. Considering the first sequence we get $y - x = 4d_1$, and considering the second sequence we obtain $y - x = 3d_2$. Thus, $4d_1 = 3d_2$. In that case,

$$\frac{b_4 - b_3}{a_2 - a_1} = \frac{2d_2}{d_1}.$$ □

Problem 13. If the function $f(x)$ is defined on $\mathbb{R}$ and satisfies both $f(10 + x) = f(10 - x)$ and $f(20 - x) = -f(20 + x)$, then $f(x)$ is

(a) periodic, but neither even nor odd (b) even and periodic (c) even, but not periodic (d) odd, but not periodic (e) odd and periodic

Answer: (e)

Solution: For any real number x, we have $f(20-x) = f(10+(10-x)) = f(10-(10-x)) = f(x)$, and $f(20 + x) = f(10 + (10 + x)) = f(10 - (10 + x)) = f(-x)$. We get $f(x) = -f(-x)$, so the function is odd. Now, $f(x+20) = -f(20 - x) = f(x - 20)$, or $f(x+20) = f(x-20)$. Set $x - 20 = y$. We obtain $f(y + 40) = f(y)$, so the function is periodic. □

Problem 14. How many solutions of the equation $\cos(7x) = \cos(5x)$ are in the interval $[0, \pi]$?

(a) 5 (b) 6 (c) 7 (d) 8 (e) 9

Answer: (c)

Solution: We have $\cos(5x) - \cos(7x) = \cos(6x - x) - \cos(6x + x) = 2\sin(6x)\sin(x)$. Therefore, the solutions in $[0, \pi]$ are the numbers $\frac{\pi}{6}k$ for $k = 0, 1, 2, 3, 4, 5, 6$. □

Problem 15. What is the radius of the inscribed circle in a triangle whose sides are 12, 13, and 5?

(a) 1 (b) $\sqrt{2}$ (c) $\sqrt{3}$ (d) 2 (e) $\sqrt{5}$

Answer: (d)

Solution: Let A, B, and C be the vertices of the triangle, so that $AB = 5$, $BC = 12$, and $AC = 13$. Also, let O be the center of the inscribed circle. Since $5^2 + 12^2 = 13^2$, $\triangle ABC$ is a right triangle. We compute its area in two different ways. First, its area is $(5 \cdot 12)/2 = 30$. Next the areas of $\triangle ABO$, $\triangle BCO$, and $\triangle ACO$ are $5r/2$, $12r/2$, and $13r/2$, respectively, so the area of $\triangle ABC$ is $30r/2$. □

Problem 16. What is the largest power of 2 that divides $13^4 - 11^4$?

(a) 8 (b) 16 (c) 32 (d) 64 (e) 128

Answer: (c)

Solution: We have $13^4 - 11^4 = (13^2)^2 - (11^2)^2 = (13^2 - 11^2)(13^2 + 11^2) = 48 \cdot 290 = 16 \cdot 3 \cdot 2 \cdot 145$. □

Problem 17. Let $F_0 = 0$ and $F_1 = 1$. For all $n \geq 2$, define F_n to be the remainder of $F_{n-1} + F_{n-2}$ divided by 3. The sequence starts as follows:

$$0, 1, 1, 2, 0, 2, \ldots$$

What is $F_{2017} + F_{2018} + F_{2019} + F_{2020} + F_{2021} + F_{2022} + F_{2023} + F_{2024}$?

(a) 6 (b) 7 (c) 8 (d) 9 (e) 10

Answer: (d)

Solution: The first few terms of the sequence are $0, 1, 1, 2, 0, 2, 2, 1, 0, 1, 1$, so the sequence is periodic with a period 8. Thus $F_{2017} = F_1$ since $2017 = 8 \cdot 252 + 1$, $F_{2018} = F_2$, etc. The sum is $1 + 1 + 2 + 0 + 2 + 2 + 1 + 0$. □

Problem 18. Consider the set of all fractions x/y where x and y are relatively prime positive integers. How many of these fractions have the property that if both the numerator and the denominator are increased by 1, the value of the fraction is increased by 10%?

(a) 0 (b) 1 (c) 2 (d) 3 (e) infinitely many.

Answer: (b)

Solution: We have $1.1x/y = (x+1)/(y+1)$ or $1.1x(y+1) = y(x+1)$. Multiplying by 10 and simplifying leads to $xy + 11x - 10y = 0$ or $11x = y(10 - x)$. Thus, y divides $11x$. Since x and y are relatively prime, y divides 11, so $y = 1$ or $y = 11$. If $y = 1$, then $x = 5/6$, not an integer. If $y = 11$, $x = 5$. □

Problem 19. Tom has 12 coins, each of which is a nickel or a dime. There are exactly 17 different values that can be obtained as combinations of one or more of his coins. How many dimes does Tom have?

(a) 3 (b) 4 (c) 5 (d) 6 (e) 7

Answer: (c)

Solution: Clearly, Tom has at least one nickel (otherwise there are exactly 12 values one can make with his coins). Thus, the values one can make are all consecutive multiples of 5 cents, starting with 5 cents and ending with the total value of Tom's coins. Since there are 17 values one can make, the total value of Tom's coins is 85 cents. If Tom has n nickels and d dimes we get the system of equations $n + d = 12$ and $5n + 10d = 17$. Solving the system we get $n = 7$ and $d = 5$. □

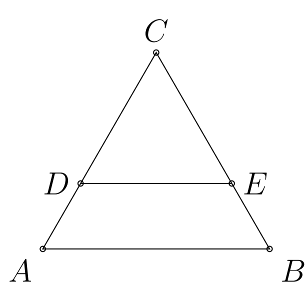
Problem 20. A regular 6-sided die with the numbers 1, 2, 3, 4, 5 and 6 is rolled twice. Next, another regular 6-sided die is rolled two times. What is the probability that the sum of the numbers rolled on the first 6-sided die is greater than the sum of the numbers rolled on the second 6-sided die?

(a) $\frac{73}{648}$ (b) $\frac{77}{648}$ (c) $\frac{77}{432}$ (d) $\frac{575}{1296}$ (e) $\frac{1}{2}$

Answer: (d)

Solution: Let S_1 be the sum of the numbers rolled on the first 6-sided die and let S_2 be the sum of the numbers rolled on the second 6-sided die. Let p_1 be the probability that $S_1 > S_2$, let p_2 be the probability that $S_1 = S_2$, and let p_3 be the probability that $S_1 < S_2$. Clearly, $p_1 + p_2 + p_3 = 1$ and $p_1 = p_3$. Therefore, $p_1 = (1/2)(1 - p_2)$. S_1 can be any integer from 2 to 12. The probability that $S_1 = k$ for k between 2 and 7 is $(k - 1)/36$; and the probability that $S_1 = k$ for k between 8 and 12 is $(13 - k)/36$. Thus, the probability that $S_1 = S_2 = k$ is $(k - 1)^2/(36)^2$ when k is between 2 and 7 ; and the probability that $S_1 = S_2 = k$ is $(13 - k)^2/(36)^2$ when k is between 8 and 12. Therefore, $p_2 = 146/1296$. □

Problem 21. In the equilateral triangle $\triangle ABC$, the point D is on $\overline{AC}$, the point E is on $\overline{BC}$, and $\overline{DE}$ is parallel to $\overline{AB}$. If the perimeter of the triangle $\triangle DEC$ is equal to the perimeter of the trapezoid $ABED$, what is the ratio of the areas of the triangle $\triangle DEC$ and the trapezoid $ABED$?

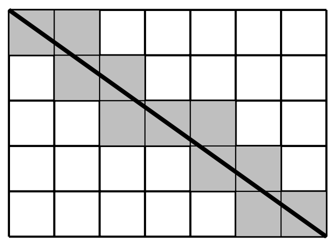


(a) $\frac{1}{3}$ (b) $\frac{9}{16}$ (c) 1 (d) $\frac{9}{7}$ (e) $\frac{16}{9}$

Answer: (d)

Solution: Let $AB = a$ and $CD = x$. Clearly, $\triangle CDE$ is equilateral and its perimeter is $3x$. The perimeter of the trapezoid $ABED$ is $a+(a-x)+x+(a-x) = 3a-x$. Thus, $3x = 3a-x$, or $x = 3a/4$. Therefore, the ratio of the areas of $\triangle DEC$ and $\triangle ABC$ is $9/16$. □

Problem 22. A diagonal of this 5×7 rectangle passes through 11 squares. (We say that a line passes through a square if the square and the line have at least two common points.)



How many squares will the diagonal of a 2016×2018 rectangle pass through?

(a) 4032 (b) 4033 (c) 4034 (d) 4035 (e) 4036

Answer: (a)

Solution: First, we show that if we consider a rectangle with integer sides k and l where k and l are relatively prime positive integers with $k < l$, then any of its diagonals passes through $k + l - 1$ squares. We can assume that the bottom left corner of the rectangle is the origin of the coordinate system $(0, 0)$ and that the the remaining three vertices of the rectangle are $(l, 0)$, (l, k), and $(0, k)$. Also, let us consider the diagonal with endpoints $(0, k)$ and $(l, 0)$. Since the absolute value of the slope of the diagonal is less than 1, for each $t = 0, 1, \dots, l-1$ the diagonal passes either through one or two of the squares which are in the strip between the vertical lines $x = t$ and $x = t + 1$. It passes through one square if the diagonal does not intersect any of the horizontal lines $y = 1,\ y = 2,\ \dots, y = k - 1$ and it passes through two squares when the diagonal does intersect one of these lines. Thus, the number of squares the diagonal passes through equals the number of vertical strips (which is k) + the number of intersections of the diagonal and the horizontal lines $y = 1,\ y = 2,\ \dots, y = k - 1$ (which is $k - 1$). So, indeed when k and l are relatively prime a diagonal passes through $k + l - 1$ squares.

Next, consider a 2016×2018 rectangle. We can assume that it has vertices $(0, 0)$, $(2018, 0)$, $(2018, 2016)$, $(0, 2016)$ and that the diagonal has endpoints $(0, 2016)$ and $(2018, 0)$. The center of the rectangle is $(1009, 1008)$. Since 1009 and 1008 are relatively prime, the diagonal passes through $1008 + 1009 - 1 = 2016$ squares of the rectangle with vertices $(0, 1008)$, $(1009, 1008)$, $(1009, 2016)$, $(0, 2016)$; it passes through $1008 + 1009 - 1 = 2016$ squares of the rectangle with vertices $(1009, 0)$, $(2018, 0)$, $(2018, 1008)$, $(1009, 1008)$; and it passes through no other squares. □

Problem 23. Suppose s and t are integers satisfying $s\sqrt{s-t} = t\sqrt{2t+s}$. If $5 \le s \le 20$ and $5 \le t \le 20$, then how many such solutions are there?

(a) 1 (b) 4 (c) 5 (d) 6 (e) 16

Answer: (d)

Solution: If we square the given equation we get $s^2(s-t) = t^2(2t+s)$ or $s^3 - s^2t - st^2 - 2t^3 = 0$. This factors as $(s-2t)(s^2+st+t^2) = 0$. Now $s^2+st+t^2 = 0$ only if $s = t = 0$. Indeed, $s^2+st+t^2 = (s+t/2)^2 + (3/4)t^2$. Thus, the equation holds only if $s = 2t$ and $s \ge 0$. Furthermore, s and t are in the given ranges exactly when $t = 5, 6, 7, 8, 9$, or 10. □

Problem 24. For each positive integer n, the parabola $y = (n^2+n)x^2 - (2n+1)x + 1$ intersects the x–axis at the points A_n and B_n. What is the value of $A_1B_1 + A_2B_2 + \cdots + A_{2018}B_{2018}$?

(a) $\frac{2017}{2019}$ (b) $\frac{2016}{2017}$ (c) $\frac{2017}{2018}$ (d) $\frac{2018}{2019}$ (e) $\frac{2018}{2017}$

Answer: (d)

Solution: The zeros of the quadratic equation $(n^2+n)x^2 - (2n+1)x + 1 = 0$ are $1/n$ and $1/(n+1)$. Thus,

$$A_nB_n = \frac{1}{n} - \frac{1}{n+1}.$$

The value of the sum is

$$\left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{2018} - \frac{1}{2019}\right) = 1 - \frac{1}{2019}.$$ □

Problem 25. For certain real numbers a, b, and c, the polynomial

$$g(x) = x^3 + ax^2 + x + 10$$

has three distinct zeros, and each zero of $g(x)$ is also a zero of the polynomial

$$f(x) = x^4 + x^3 + bx^2 + 100x + c.$$

What is $f(1)$?

(a) -9009 (b) -8008 (c) -7007 (d) -6006 (e) -5005

Answer: (c)

Solution: Since g has distinct zeros and each zero of $g(x)$ is a zero of $f(x)$, then $g(x)$ divides $f(x)$. Write $f(x) = q(x) \cdot g(x) + r(x)$ with quotient $q(x)$ and remainder $r(x)$ given by

$$q(x) = x + (1-a), \qquad r(x) = \big(b - 1 - a(1-a)\big)x^2 + \big(90 - (1-a)\big)x + \big(c - 10(1-a)\big).$$

Thus, $b - 1 - a(1-a) = 0$, $90 - (1-a) = 0$, and $c - 10(1-a)$. Therefore, $a = -89$, $b = -8009$, and $c = 900$. □

Problem 26. Suppose x, y and z are three real numbers such that $3x$, $4y$, $5z$ is a geometric sequence and $1/x$, $1/y$, $1/z$ is an arithmetic sequence. What is the value of $(x/z) + (z/x)$?

(a) 32/15 (b) 34/15 (c) 37/15 (d) 38/15 (e) 64/15

Answer: (b)

Solution: Since, $3x$, $4y$, $5z$ is a geometric sequence, $(4y)/(3x) = (5z)/(4y)$, so $16y^2 = 15xz$. Also, since $1/x, 1/y, 1/z$ is an arithmetic sequence,

$$\frac{1}{y} - \frac{1}{x} = \frac{1}{z} - \frac{1}{z},$$

or

$$\frac{2}{y} = \frac{1}{x} + \frac{1}{z} = \frac{x+z}{xz}.$$

On the other hand,

$$\frac{x}{z} + \frac{z}{x} = \frac{x^2+z^2}{xz} = \frac{(x+z)^2}{xz} - 2 = \frac{4xz}{y^2} - 2 = \frac{64}{15} - 2. \qquad \square$$

Problem 27. If a_1 and a_2 are positive integers with $a_1 > a_2$ we define a sequence of integers as follows: $a_{n+2} = |a_{n+1} - a_n|$ for $n = 1, 2, \ldots$ if $a_n > 0$ and $a_{n+1} > 0$, and we stop when $a_{n+1} = 0$. For example, if $a_1 = 10$ and $a_2 = 6$, we get the sequence $10, 6, 4, 2, 2, 0$.

If we start with positive integers a_1 and a_2, both between 1 and 10, what is the length of the longest sequence one can obtain? (The length of a sequence is the number of terms in the sequence, for example the sequence $10, 6, 4, 2, 2, 0$ has length 6).

(a) 6 (b) 7 (c) 9 (d) 10 (e) 17

Answer: (e)

Solution: If $a_1 = 1$ and $a_2 = 10$, we get the sequence $1, 10, 9, 1, 8, 7, 1, 6, 5, 1, 4, 3, 1, 2, 1, 0$. This is the longest sequence one may get starting with two numbers between 1 and 10.

One can prove something more general. If $m \geq 2$, let $l(m)$ be the length of the longest sequence one can construct using the given rules, ($a_{n+2} = |a_{n+1} - a_n|$ if $a_{n+1} > 0$, and stop if $a_{n+1} = 0$), provided we start with two numbers between 1 and m. Define $u(m) = 3k + 2$ if m is even and $m = 2k$; and define $u(m) = 3k + 4$ if m is odd and $m = 2k + 1$.

We claim that $l(m) = u(m)$ for all $m \geq 3$. It can be checked directly that $l(2) = u(2) = 5$ and $l(3) = u(3) = 7$. Also, taking $a_1 = 1$, $a_2 = m$ or $a_1 = m - 1$, $a_2 = m$ produces a sequence with length $u(m)$. What remains is to show that $l(m) \leq u(m)$ for all $m \geq 3$.

Next, let $m > 4$. Assume that we have proved that $l(m) \leq u(m)$ for all integers between 3 and $m - 1$. Let a_1 and a_2 be integers between 1 and m. Denote by $L = L(a_1, a_2)$ the length of the sequence we get by starting with the given numbers a_1 and a_2. We need to show $L \leq u(m)$. If $a_1 < m$ and $a_2 < m$, then $L \leq l(m-1) \leq u(m-1) < u(m)$.

Next, suppose $a_1 = m$, $a_2 = l < m$. Then, the sequence starts $m, l, m - l$, so $L \leq 1 + l(m-1) \leq 1 + u(m-1) \leq u(m)$.

Next, let $a_1 = l < m$, and $a_2 = m$. First, suppose $l \neq 1$ and $l \neq m-1$. Then the sequence starts $l, m, m-l, l$, so $L \leq 2 + l(m-2) \leq 2 + u(m-2) \leq u(m)$.

Finally, if $l = 1$ or $l = m-1$, direct computation shows that $L \leq u(m)$. □

Problem 28. Many high schools in South Carolina participated in a math contest. Each such high school sent in a team of three contestants. Suppose that each contestant earned a different score. The contestant that scored the median among all students had the highest score on their team while their teammates placed 59^{th} and 106^{th}. How many schools participated in the contest?

(a) 36 (b) 37 (c) 38 (d) 39 (e) There is not enough information to determine the number

Answer: (b)

Solution: Given that the student who was 59^{th} was below the median, there were less than 117 participants, that is, less than 39 high schools participated in the contest. Also, there were at least 106 participants, so at least 36 high schools participated in the contest. Finally, the number of participants is an odd number, since all scores where distinct and one equaled the median. Therefore, an odd number of high schools participated. □

Problem 29. How many different ways can one order the integers 1 through 5 so that no three consecutive integers in the ordering are in increasing order?

(a) 60 (b) 64 (c) 70 (d) 72 (e) 120

Answer: (c)

Solution: Let A_1 be the set of all permutations $(a_1, a_2, a_3, a_4, a_5)$ of the integers $1, 2, 3, 4, 5$ such that $a_1 < a_2 < a_3$; let A_2 be the set of all permutations $(a_1, a_2, a_3, a_4, a_5)$ of $1, 2, 3, 4, 5$ such that $a_2 < a_3 < a_4$; and let A_3 be the set of all permutations $(a_1, a_2, a_3, a_4, a_5)$ of the integers $1, 2, 3, 4, 5$ such that $a_3 < a_4 < a_5$. We need to count how many permutations of $1, 2, 3, 4, 5$ are not in the union of the sets A_1, A_2, and A_3.

First, there are $5! = 120$ permutations of $1, 2, 3, 4, 5$.

Next, by inclusion-exclusion

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|.$$

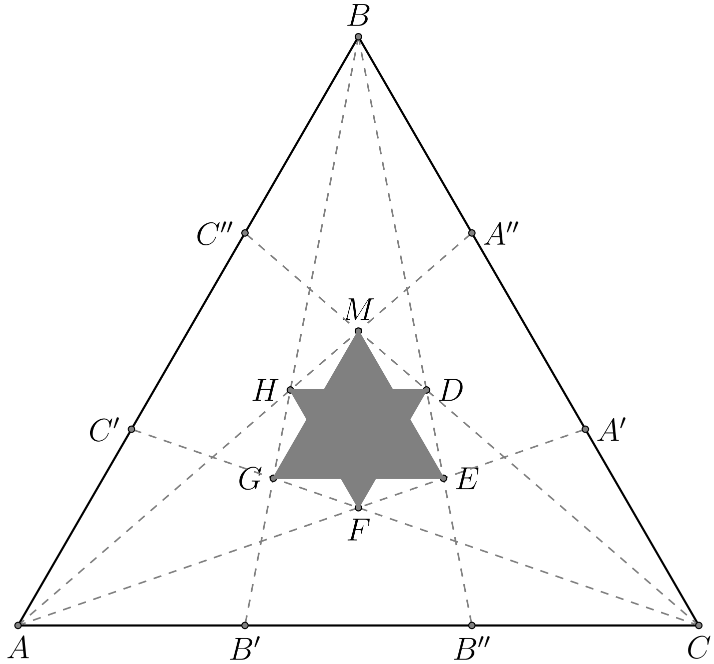
We have $|A_1| = |A_2| = |A_3| = \binom{5}{3} \cdot 2 = 20$ (for example, to compute $|A_1|$ note that there are $\binom{5}{3}$ ways to choose a_1, a_2, and a_3, and two ways to pick a_4 out of the remaining two numbers).

Also, $A_1 \cap A_2$ is the set of all permutations of $1, 2, 3, 4, 5$ where $a_1 < a_2 < a_3 < a_4$, there are 5 such permutations (there are 5 choices for a_5). Similarly, $A_2 \cap A_3$ is the set of all permutations of $1, 2, 3, 4, 5$ where $a_2 < a_3 < a_4 < a_5$, there are 5 such permutations.

Finally, $A_1 \cap A_3$ and $A_1 \cap A_2 \cap A_3$ is the set of all permutations of $1, 2, 3, 4, 5$ where $a_1 < a_2 < a_3 < a_4 < a_5$, that is the set $\{(1, 2, 3, 4, 5)\}$.

Thus, $|A_1 \cup A_2 \cup A_3| = 20 + 20 + 20 - 5 - 1 - 5 + 1 = 50$. □

Problem 30. Given an equilateral triangle $\triangle ABC$, consider the points A' and A'' that trisect side $\overline{BC}$, B' and B'' that trisect side $\overline{AC}$, and C' and C'' that trisect side $\overline{AB}$ as shown. What percentage of the area of $\triangle ABC$ is the area of the shaded star?



(a) 6% (b) 7% (c) 8% (d) 9% (e) 10%

Answer: (b)

Solution: Draw a coordinate system, so that the x–axis is along $\overline{AC}$, the origin is the midpoint of $\overline{AC}$, $AC = 2$, and the triangle is in the upper half-plane. Then, A, B, and C have coordinates $(-1, 0)$, $(0, \sqrt{3})$, and $(1, 0)$ respectively. The area of $\triangle ABC$ is $\sqrt{3}$.

Note that the whole configuration is invariant under rotations by 120° with respect to the center of the triangle. Also, it is invariant with respect to a reflection with respect to the y–axis.

Thus, $\triangle EGM$ is equilateral, and so is $\triangle DFH$. Moreover, $\overline{GE}$ and $\overline{HD}$ are parallel to the x–axis.

The coordinates of the points A', A", B', B", C', C" are af follows:

$$A' = \left(\tfrac{2}{3}, \tfrac{\sqrt{3}}{3}\right), \qquad A'' = \left(\tfrac{1}{3}, 2\tfrac{\sqrt{3}}{3}\right),$$
$$B' = \left(-\tfrac{1}{3}, 0\right), \qquad B'' = \left(\tfrac{1}{3}, 0\right),$$
$$C' = \left(-\tfrac{2}{3}, \tfrac{\sqrt{3}}{3}\right) \qquad C'' = \left(-\tfrac{1}{3}, 2\tfrac{\sqrt{3}}{3}\right).$$

Computing the equations of segments $\overline{CC'}$ and $\overline{BB'}$ we get that their point of intersection G has coordinates $(-1/4, \sqrt{3}/4)$. By symmetry, E has coordinates $(1/4, \sqrt{3}/4)$. Thus, the equilateral triangle $\triangle EGM$ has has side length $1/2$ and area $\sqrt{3}/16$.

Denote by P, Q, R, S, T, and U the remaining six vertices of the star, so that P and Q are on $\overline{EG}$, R and S are on $\overline{GM}$, and T and U are on $\overline{ME}$.

Since $\triangle EGM$ and $\triangle DFH$ are equilateral and $\overline{GE}$ and $\overline{HD}$ are parallel, $\triangle PQF$ is also an equilateral triangle. Because of the rotational invariance, $\triangle PQF$, $\triangle RSH$, and $\triangle TUD$, are congruent triangles. Thus, to find the area of the star we only need to find the area of $\triangle PQF$.

Using the equation of segment $\overline{CC'}$ and the fact the F has x–coordinate 0 we get that F has coordinates $(0, \sqrt{3}/5)$. Thus, the height of the equilateral triangle $\triangle PQF$ is $\sqrt{3}/20$, and the area of $\triangle PQF$ is $\sqrt{3}/400$. Therefore, the area of the star is $\sqrt{3}/16+3\sqrt{3}/400 = 7\sqrt{3}/100$. $\square$